Cellular homology of S^2

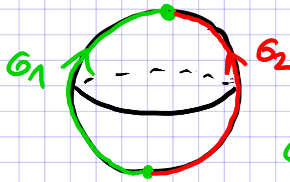
2 0-cells : S (south pole), N (north pole)

3 1-cells : σ_1, σ_2 line segments "from S to N"

$f_{\partial\sigma_1}, f_{\partial\sigma_2} : \{0,1\} \to \{S,N\}$

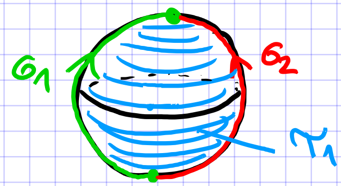
$0 \longmapsto S$

$1 \longmapsto N$

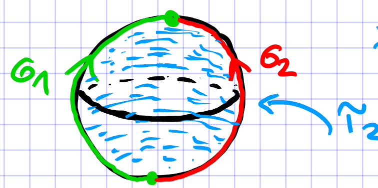


2 2-cells : τ_1 "the front face"

τ_2 "the back face"



$f_{\partial\tau_1} : S^1 \to K_{\sigma_1} \cup K_{\sigma_2}$



$f_{\partial\tau_2} : S^1 \to K_{\sigma_1} \cup K_{\sigma_2}$

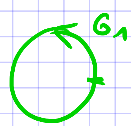
(You could also use different attaching maps, e.g. $f_{\partial\tau_1} = f_{\partial\tau_2}$. I just picked one possibility.)

Choose π_{σ_j} so that

$$\begin{tikzcd} I^1 \arrow[r, "f_{\sigma_j}"] \arrow[rrr, bend right, "\gamma_1"'] & K^1 \arrow[r] & K^1/K^0 \arrow[r, "\pi_{\sigma_j}"] & S^1 \end{tikzcd}$$

So π_{σ_j} is so that

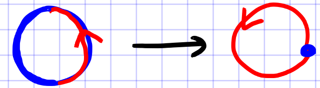
$$K^1/K^0 \xrightarrow{\pi_{\sigma_1}} S^1$$



$$K^1/K^0 \xrightarrow{\pi_{\sigma_2}} S^1$$

Compute

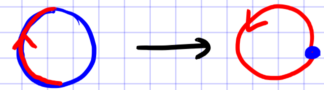
$$d\tau_1 = [\sigma_1 : \tau_1]\,\sigma_1 + [\sigma_2 : \tau_1]\,\sigma_2$$

$$[\sigma_1 : \tau_1] = \deg\left(\pi_{\sigma_1} \circ f_{\partial \tau_1}\right) = \deg\Big($$  $$\Big) = 1$$

$$[\sigma_2 : \tau_1] = \deg\left(\pi_{\sigma_2} \circ f_{\partial \tau_1}\right) = \deg\Big($$  $$\Big) = -1$$

$$\Longrightarrow \quad d\tau_1 = \sigma_1 - \sigma_2$$

$$d\tau_2 = [\sigma_1 : \tau_2]\,\sigma_1 + [\sigma_2 : \tau_2]\,\sigma_2$$

$$[\sigma_1 : \tau_2] = \deg\left(\pi_{\sigma_1} \circ f_{\partial \tau_2}\right) = \deg\Big($$  $$\Big) = -1$$

$$[\sigma_2 : \tau_2] = \deg\left(\pi_{\sigma_2} \circ f_{\partial \tau_2}\right) = \deg\Big($$  $$\Big) = +1$$

$$\Longrightarrow \quad d\tau_2 = \sigma_2 - \sigma_1$$

$$\Longrightarrow \quad 0 \longrightarrow \mathbb{Z}\tau_1 \oplus \mathbb{Z}\tau_2 \xrightarrow[\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}]{d_2} \mathbb{Z}\sigma_1 \oplus \mathbb{Z}\sigma_2 \xrightarrow[\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}]{d_1} \mathbb{Z}\{S\} \oplus \mathbb{Z}\{N\} \longrightarrow 0$$

see next page

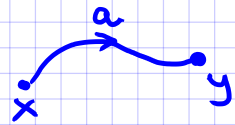
$$\ker d_2 = \langle \tau_1 + \tau_2 \rangle \qquad\qquad H_2(S^2) \cong \mathbb{Z}\langle \tau_1 + \tau_2 \rangle$$

$$\operatorname{Im} d_2 = \langle \sigma_2 - \sigma_1 \rangle = \ker(d_1) \quad \Longrightarrow \quad H_1(S^2) \cong 0$$

$$\operatorname{Im} d_1 = \langle N - S \rangle \qquad\qquad H_0(S^2) \cong \mathbb{Z}\langle N - S \rangle$$

<u>Exc</u> Compute the cellular homology using other CW-structures.

Computation of d_1 K CW-complex



x, y 0-cells in K, a 1-cell with

$$\begin{aligned} f_{\partial a} : \partial I^1 = \{0,1\} &\longrightarrow \{x,y\} \subset K^{(0)} \\ 0 &\longmapsto x \\ 1 &\longmapsto y \end{aligned}$$

What is $\quad da = [y:a]y + [x:a]x$?

$$\begin{aligned} p_x : K^{(0)} \longrightarrow K^{(0)}/K^{(-1)} = K^{(0)}/\phi = K^{(0)} \cup \{*\} = \{*,x,y\} &\longrightarrow S^0 = \{0,1\} \\ *, y &\longmapsto 0 \\ x &\longmapsto 1 \end{aligned}$$

and so: $p_x(x) = 1$, $p_x(y) = 0$.

We compute

$$[x:a] = \deg(p_x f_{\partial a}) = \deg\begin{pmatrix} \{0,1\} \longrightarrow K^{(0)} \longrightarrow \{0,1\} \\ 0 \longmapsto x \longmapsto 1 \\ 1 \longmapsto y \longmapsto 0 \end{pmatrix} = -1 .$$

Similarly,

$$\begin{aligned} p_y : K^{(0)} \longrightarrow K^{(0)}/K^{(-1)} = K^{(0)}/\phi = K^{(0)} \cup \{*\} = \{*,x,y\} &\longrightarrow S^0 = \{0,1\} \\ *, x &\longmapsto 0 \\ y &\longmapsto 1 \end{aligned}$$

and so $\quad p_y(x) = 0$, $p_y(y) = 1$.

We compute

$$[y:a] = \deg(p_y f_{\partial a}) = \deg\begin{pmatrix} \{0,1\} \longrightarrow K^{(0)} \longrightarrow \{0,1\} \\ 0 \longmapsto x \longmapsto 0 \\ 1 \longmapsto y \longmapsto 1 \end{pmatrix} = +1$$

$$\Longrightarrow \boxed{da = y - x}$$

Topology on CW-complex

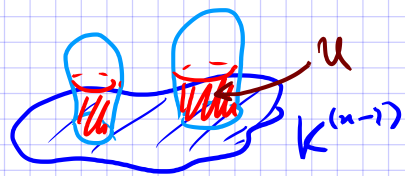
$K = \bigcup K^{(n)}$ CW complex.

$\mathcal{U} \subset K$ open $\iff$ $\mathcal{U} \cap K^{(n)} \subset K^{(n)}$ open for all n

$\iff$ $f_\sigma^{-1}(\mathcal{U}) \subset B_\sigma^n$ open for all cells σ

So for example,

$$W = K^{(n-1)} \bigcup_{\sigma \in I_n} f_\sigma(\mathcal{U}) \quad \text{for } \mathcal{U} \subset B_\sigma^n \text{ open}$$



is open in $K^{(n)}$ because:

$\forall \sigma \in I_k, k \leq n-1$: $f_\sigma^{-1}(W) = B_\sigma^k \subset B_\sigma^k$ is open

$\forall \sigma \in I_n$: $f_\sigma^{-1}(W) = \mathcal{U} \subset B_\sigma^n$ is open.

<u>Excision</u> (X,A) pair of spaces, $B \subset A$ any subset s.th. $cl(B) \subset int(A)$.

Then $i: (X\setminus B, A\setminus B) \longrightarrow (X,A)$ induces an iso in homology.

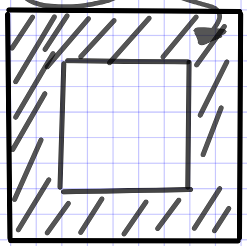
<u>Rmk</u> B is not assumed to be open. See lecture #12A, Theorem 2.

Lecture #9A, page 9:

$$\begin{tikzcd}
H_*(Z^n, \partial Z^n) \arrow[r, "f_*"] \arrow[d, "inc_*"', "\cong"] & H_*(K^{(n)}, K^{(n-1)}) \arrow[d, "\cong"] \\
H_*\Big(\coprod_{\sigma \in I_n} (I_\sigma^n, I_\sigma^n \setminus \{0\})\Big) & H_*\Big(K^{(n)}, K^{(n)} \setminus \bigcup_{\sigma\in I_n} f_\sigma(0)\Big) \\
H_*\Big(\coprod_{\sigma\in I_n} (Int\, I^n_{1-\varepsilon}, Int\, I^n_{1-\varepsilon}\setminus\{0\})\Big) \arrow[r, "\cong"] \arrow[u, "inc_*", "\cong"', "(*)"] & H_*\Big(K^{(n)} \setminus K_\varepsilon^{(n-1)}, K^{(n)}\setminus K_\varepsilon^{(n-1)} \setminus \bigcup_{\sigma\in I_n} f_\sigma(0)\Big) \arrow[u, "inc_*"', "\cong \text{ by excision}", "(*)"']
\end{tikzcd}$$

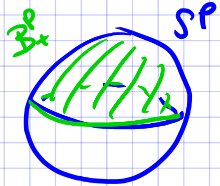
(*) Here, apply excision as above.

$$K_\varepsilon^{(n-1)} := K^{(n-1)} \cup \bigcup_{\sigma\in I_n} f_\sigma(\text{the grey shell})$$



Alternatively, you can work with $I^n_{1-\varepsilon}$ instead of $Int\, I^n_{1-\varepsilon}$ and the interior of the grey shell instead of the grey shell. Then you can use the weaker version of excision as given in the axioms for homology.

Existence of diagram :

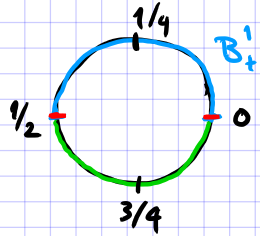


$$\begin{tikzcd}
(B^p_+, S^{p-1}) \arrow[r, hook, "i"] \arrow[d, "\approx"] & (S^p, B^p_-) \arrow[d, "h"] \\
(I^p, \partial I^p) \arrow[r, "\gamma_p"] & (S^p, *)
\end{tikzcd}$$

(Lecture #9A, page 4)

$p = 1$:

$$\begin{tikzcd}
(B^1_+, S^0) \arrow[r, hook, "i"] \arrow[d, "f"] & (S^1, B^1_-) \arrow[d, "h"] \\
(I^1 = [0,1], \partial I^1) \arrow[r, "\gamma_1"'] & \left(I^1/\partial I^1, *\right) \approx (S^1, *)
\end{tikzcd}$$



$$S^1 \approx [0,1]/_{0 \sim 1} \approx \mathbb{R}/\mathbb{Z}$$

$$\left.\begin{array}{ll} i : B^1_+ = [-1,1] \longrightarrow S^1, & i(x) = \frac{x}{4} + \frac{1}{4} \mod 1 \\ h : S^1 \longrightarrow S^1, & h(x) = 2x \mod 1 \end{array}\right\} \Rightarrow h \circ i(x) = \frac{x+1}{2} \mod 1$$

$$\left.\begin{array}{ll} \gamma_1 : I^1 \longrightarrow S^1, & \gamma_1(x) = x \mod 1 \\ f : B^1_+ \xrightarrow{\approx} I^1, & f(x) = \frac{x+1}{2} \end{array}\right\} \Rightarrow \gamma_1 \circ f(x) = \frac{x+1}{2} \mod 1$$

Hint for $p>1$:

$$
\begin{tikzcd}
(B^p_+, S^{p-1}) \arrow[r, hook, "i"] \arrow[d, phantom, "\cong"] & (S^p, B^p_-) \arrow[d, "h"] \\
(I^p, \partial I^p) \arrow[r, "\gamma_p"] & (S^p, *)
\end{tikzcd}
$$

$$
\begin{tikzcd}
(B^p_+, S^{p-1}) \arrow[rr, "i"] & & (S^p, B^p_-) \arrow[dddr, bend left=40, "h"] & \\
\cong & \text{to show.} & \cong & \text{to show} \\
(B^1_+ \times \cdots \times B^1_+, \partial(B^1_+ \times \cdots \times B^1_+)) \arrow[r, "i\times\cdots\times i"] & (S^1 \times \cdots \times S^1, (i\times\cdots\times i)(\partial(B^1_+\times\cdots\times B^1_+))) \arrow[r, "\pi"] \arrow[d, "h\times\cdots\times h"] & (S^1 \vee \cdots \vee S^1, *) \arrow[d, "h\vee\cdots\vee h"] & \\
(I^p, \partial I^p) \arrow[u, "f^{-1}\times\cdots\times f^{-1}"] \arrow[r, "\gamma\times\cdots\times\gamma"] \arrow[rr, bend right=30, "\gamma_p"'] & (S^1 \times \cdots \times S^1, (\gamma\times\cdots\times\gamma)(\partial I^p)) \arrow[r, "\pi"] & (S^1 \vee \cdots \vee S^1, *) &
\end{tikzcd}
$$

Bifunctor: Given categories $\mathcal{C}, \mathcal{C}', \mathcal{D}$, a bifunctor F is a functor

$$F : \mathcal{C} \times \mathcal{C}' \longrightarrow \mathcal{D},$$

where $\mathcal{C} \times \mathcal{C}'$ is the product category. Concretely, F consists of assignments

$$\forall\, X \in \mathrm{Ob}\,\mathcal{C},\ Y \in \mathrm{Ob}\,\mathcal{C}' : \quad F(X,Y) \in \mathrm{Ob}(\mathcal{D})$$

$$\forall\, f \in \mathrm{Mor}_{\mathcal{C}}(X,X'),\ g \in \mathrm{Mor}_{\mathcal{C}'}(Y,Y') : \quad F(f,g) \in \mathrm{Mor}_{\mathcal{D}}\big(F(X,Y), F(X',Y')\big)$$

such that

* $F(\mathrm{id}_X, \mathrm{id}_Y) = \mathrm{id}_{F(X,Y)}$

* $F(f' \circ f, g' \circ g) = F(f',g') \circ F(f,g)$

Wedge Product as bifunctor:

$$\vee : \quad \mathrm{Top}_* \times \mathrm{Top}_* \longrightarrow \mathrm{Top}_*$$

where Top_* is the category of pointed topological spaces with continuous maps preserving basepoints.

$\vee$ consists of

$\forall\ X, Y$ pointed spaces $\rightsquigarrow$ $X \vee Y$ pointed space

$\forall\ f : X \to X',\ g : Y \to Y'$ $\rightsquigarrow$ $f \vee g : X \vee Y \longrightarrow X' \vee Y'$

It satisfies

$$id_X \vee id_Y = id_{X \vee Y}$$

and $\forall\ X \xrightarrow{f} X' \xrightarrow{f'} X'',\ \forall\ Y \xrightarrow{g} Y' \xrightarrow{g'} Y''$:

$$\begin{tikzcd}
X \vee Y \arrow[r, "f \vee g"] \arrow[rr, bend right, "(f' \circ f) \vee (g' \circ g)"'] & X' \vee Y' \arrow[r, "f' \vee g'"] \arrow[d, phantom, "\circlearrowleft"] & X'' \vee Y'' \\
& {} &
\end{tikzcd}$$

So $\vee$ is a bifunctor.

Similar for smash product.