An alternative solution to Exercise 11

Prop Let G be a compact topological group acting transitively on the compact Hausdorff space X.
If X and $\mathrm{Stab}_G(x_0)$ are connected, then G is connected.
Here:

$$\mathrm{Stab}_G(x_0) = \{g \in G \mid g \cdot x_0 = x_0\}, \quad x_0 \in X.$$

Proof Consider the surjective continuous map

$$\Theta : G \longrightarrow X, \quad g \longmapsto g x_0 .$$

For $x \in X$, choose $g \in G$ with $g x_0 = x$. Then

$$\Theta^{-1}(\{x\}) = g\,\mathrm{Stab}_G(x_0)$$

$\mathrm{Stab}_G(x_0) = \Theta^{-1}(\{x_0\}) \subset G$ is a closed subgroup, hence it is compact.
Therefore, the bijective continuous map

$$\phi : \mathrm{Stab}_G(x_0) \longrightarrow g\,\mathrm{Stab}_G(x_0)$$

is a homeomorphism and so $\Theta^{-1}(\{x\}) = g\,\mathrm{Stab}_G(x_0)$ is connected for every $x \in X$. Now elementary arguments show that G is connected:

Suppose $G = \mathcal{U} \cup V$, $\mathcal{U} \cap V = \phi$, $\mathcal{U}, V \subset G$ open.
Then

$$\Theta^{-1}(x) = \left(\Theta^{-1}(x) \cap \mathcal{U}\right) \cup \left(\Theta^{-1}(x) \cap V\right).$$

Connectedness of $\Theta^{-1}(x)$ implies that

$$\Theta^{-1}(x) \subseteq \mathcal{U} \quad \text{or} \quad \Theta^{-1}(x) \subseteq V.$$

Thus

$$X = \Theta(\mathcal{U}) \cup \Theta(V)$$

$$\Theta(\mathcal{U}) \cap \Theta(V) = \phi$$

$\Theta(\mathcal{U}), \Theta(V) \subset X$ are open.

$$\left(\begin{array}{l} G \setminus \mathcal{U} \text{ closed} \\ \overset{G}{\underset{\text{compact}}{\Longrightarrow}} G \setminus \mathcal{U} \text{ compact} \\ \Longrightarrow X \setminus \Theta(\mathcal{U}) = \Theta(X \setminus \mathcal{U}) \subset X \\ \qquad \text{is compact} \\ \overset{X \text{ Haus-}}{\underset{\text{dorff}}{\Longrightarrow}} X \setminus \Theta(\mathcal{U}) \text{ closed} \end{array}\right)$$

X is connected, hence $\Theta(\mathcal{U}) = \phi$ or $\Theta(V) = \phi$.

Thus $\mathcal{U} = \phi$ or $V = \phi$ and so G is connected.

$\square$

* $\underline{SO(n)}$ acts transitively on S^n via matix multiplication

$$\begin{aligned} SO(n+1) &\longrightarrow S^n \\ A &\longmapsto Ae_1 \end{aligned}$$

Here, we view

$$SO(n+1) \subset \mathbb{R}^{(n+1)\times(n+1)}$$

$$S^n \subset \mathbb{R}^{n+1}.$$

One shows

$$\mathrm{Stab}_{SO(n+1)}(e_1) \cong SO(n).$$

By the previous proposition,

$$SO(n) \text{ connected} \Rightarrow SO(n+1) \text{ connected}.$$

Induction (and the base case $SO(2) \cong S^1$) implies that $SO(n)$ is connected for all $n \geq 2$.

As $SO(n)$ is "locally euclidean", path connectedness follows from connectedness.

* **$SU(n)$** acts transitively on $S^{2n-1} \subset \mathbb{C}^n$.

  The stabilizer is diffeomorphic to $SU(n-1)$.
  The same proof as before works.

* **$U(n)$** acts transitively on $S^{2n-1} \subset \mathbb{C}P^n$, with stabilizer $U(n-1)$.

  The same proof applies again.

* $Gl(n,\mathbb{C})$, $Gl^+(n,\mathbb{R})$ are non-compact. See the solutions for these groups.

An explanation for the last conclusion in the proof of the 1st corollary in #8A

Last line of the proof:

$$\deg(\phi\circ f) = \sum_{j=1}^{k} \varepsilon_{q_j}(\phi\circ f) = \sum_{j=1}^{k} \varepsilon_{q_j}(f).$$

First equality: $\phi\circ f$ satisfies the assumption for the last theorem in lecture #7B. Therefore, $(\phi\circ f)_* = \sum_{j=1}^{k} (\phi\circ f)_{j,*}$ where

$$(\phi\circ f)_j(x) = \begin{cases} \phi(f(x)) & \text{for } x\in B_j \\ -p & \text{else} \end{cases}.$$

Hence,

$$\deg(\phi\circ f) = \sum_{j=1}^{k} \deg((\phi\circ f)_j).$$

Note that $(\phi\circ f)_j$ has p as a regular value with $(\phi\circ f)_j^{-1}(p)=\{q_j\}$.

Proposition in #7B $\Longrightarrow$ $\deg(\phi\circ f)_j = \varepsilon_{q_j}((\phi\circ f)_j)$.

Finally, $\varepsilon_{q_j}((\phi\circ f)_j) = \varepsilon_{q_j}(\phi\circ f)$ because the local degree only depends on the map near q_j and $\phi\circ f$ and $(\phi\circ f)_j$ coincide on B_j.

We conclude:

$$\deg(\phi\circ f) = \sum_{j=1}^{k} \varepsilon_{q_j}(\phi\circ f).$$

Second equality: Let $\sigma \in SO(n)$ with $\sigma(p) = q_j$. Then

$$\begin{aligned}
\varepsilon_{q_j}(\phi \circ f) &= \operatorname{sgn} \det \left(D(\sigma \circ \phi \circ f)_{q_j} \right) \\
&= \operatorname{sgn} \det \Big(D\sigma_p \underbrace{D\phi_p}_{= \mathrm{Id} \text{ (by construction of } \phi)} Df_{q_j} \Big) \\
&= \operatorname{sgn} \det \left(D(\sigma \circ f)_{q_j} \right) \\
&= \varepsilon_{q_j}(f)
\end{aligned}$$