1) Solution to Ex 3 suggested by a student:

Let X be path-connected, locally path-connected and semilocally simply connected. Suppose that $G = \pi_1(X)$ is not abelian but solvable. Then by Prop 1.36 in Hatcher, there exists a covering space $p: Y \longrightarrow X$ so that

$$p_\#(\pi_1(Y)) = [G,G] \neq \{0\}.$$

Consider the diagram

$$\begin{tikzcd} \pi_1(Y) \arrow[r, "\phi_Y"] \arrow[d, "p_\#"'] \arrow[dr, phantom, "///"] & H_1(Y) \arrow[d, "p_*"] \\ G = \pi_1(X) \arrow[r, "\phi_X"] & H_1(X) \end{tikzcd}$$

We get that

$$p_*\Big(\underbrace{\phi_Y(\underbrace{\pi_1(Y)}_{\text{not abelian}})}_{\neq\{0\}}\Big) = \phi_X \circ p_\#(\pi_1(Y)) = \phi_X([G,G]) = 0.$$

This shows that $\ker(p_*) \neq \{0\}$, hence p_* is not injective.

An example for such a space X is $\mathbb{R}P^2 \vee \mathbb{R}P^2$:

$$\pi_1(\mathbb{R}P^2 \vee \mathbb{R}P^2) \underset{\substack{\uparrow \\ \text{Ex 5}}}{\cong} \pi_1(\mathbb{R}P^2) * \pi_1(\mathbb{R}P^2) \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$$

A resolution of the non-abelian group $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ is

$$\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} \supset \{\text{"words of even length"}\} \supset \{0\}$$

2) Any group is the fundamental group of a space.

Let G be a group. G can be represented as a quotient of a free group F:

$$G \cong F / \langle \{f_i, i \in I\} \rangle \quad , \quad \{f_i, i \in I\} \subset F .$$

Define X^1 to be a bouquet of circles with one circle for every generator of F.
Each word f_i corresponds to a loop γ_i in X^1. For each word $f_i \in F$, attach a disk D^2 along the loop γ_i.
The resulting 2-dimensional space X satisfies

$$\pi_1(X) \cong G .$$

3) <u>LES for pairs</u>: let $A \subset X$. There are two versions:

* The SES $0 \to S_\cdot(A) \longrightarrow S_\cdot(X) \longrightarrow S_\cdot(X, A) \longrightarrow 0$ induces a LES

$$\boxed{\dots \to H_n(A) \longrightarrow H_n(X) \to H_n(X, A) \longrightarrow H_{n-1}(A) \longrightarrow \dots}$$

* <u>Reduced version</u>:

The SES $0 \to \tilde{S}_\cdot(A) \longrightarrow \tilde{S}_\cdot(X) \longrightarrow S_\cdot(X, A) \longrightarrow 0$ induces a LES

$$\boxed{\dots \longrightarrow \tilde{H}_n(A) \longrightarrow \tilde{H}_n(X) \longrightarrow H_n(X, A) \longrightarrow \tilde{H}_{n-1}(A) \longrightarrow \dots}$$

Here: $\tilde{S}_*(X) = \big(\dots \to S_2(X) \to S_1(X) \to \underset{\substack{\uparrow \\ \deg = 0}}{S_0(X)} \xrightarrow{\varepsilon} \mathbb{Z} \to 0 \big)$