Additivity axiom in reduced homology

Suppose that $X = A \amalg B$ and $A \neq \emptyset \neq B$.

Additivity axiom: $\quad H_*(X) \cong H_*(A) \oplus H_*(B)$.

This does not hold for reduced homology, but we have a SES

$$0 \longrightarrow \tilde{H}_0(A) \oplus \tilde{H}_0(B) \xrightarrow{(i_A)_* \oplus (i_B)_*} \tilde{H}_0(X) \longrightarrow \mathbb{Z} \longrightarrow 0.$$

Proof

- Consider the blue diagram below. It commutes.
- Consider the top row, where the first map is induced from $(i_A)_* \oplus (i_B)_*$ and the second map $\tilde{H}_0(X) \longrightarrow \mathbb{Z}$ is the composition of $(p_A)_* : \tilde{H}_0(X) \to H_0(A)$ with ε_A.
- It's straight forward to check that the top row is a SES.

$$\begin{tikzcd}
 & 0 \arrow[d] & & 0 \arrow[d] & & \\
0 \arrow[r] & \tilde{H}_0(A) \oplus \tilde{H}_0(B) \arrow[rr, "{([a],[b]) \mapsto [a+b]}"] \arrow[d] & & \tilde{H}_0(X) \arrow[rr] \arrow[d] \arrow[dr, "(p_A)_*"'] & & \mathbb{Z} \arrow[r] & 0 \\
 & H_0(A) \oplus H_0(B) \arrow[rr, "\cong"] \arrow[d, "\varepsilon_A \oplus \varepsilon_B"] & & H_0(X) \arrow[d, "\varepsilon_X"] & H_0(A) \arrow[ur, "\varepsilon_A"'] & & \\
 & \mathbb{Z} \oplus \mathbb{Z} \arrow[rr, "+"'] \arrow[d] & & \mathbb{Z} \arrow[d] & & & \\
 & 0 & & 0 & & &
\end{tikzcd}$$

$\square$

The "additivity" formula for reduced homology shows that reduced Mayer-Vietoris also works for $A \cap B = \emptyset$ if $X = A \amalg B$ and $A \neq \emptyset \neq B$:

$$\cdots \longrightarrow 0 = H_n(A \cap B) \longrightarrow H_n(A) \oplus H_n(B) \xrightarrow{\cong} H_n(X) \longrightarrow$$

$$\longrightarrow 0 = H_{n-1}(A \cap B) \longrightarrow \cdots$$

$$\vdots$$

$$\cdots \longrightarrow H_1(X) \longrightarrow$$

$$\longrightarrow 0 = \tilde{H}_0(A \cap B) \longrightarrow \tilde{H}_0(A) \oplus \tilde{H}_0(B) \longrightarrow \tilde{H}_0(X) \longrightarrow \underset{\substack{\| \\ H_{-1}(A \cap B) \\ \| \\ H_{-1}(\emptyset)}}{\mathbb{Z}} \longrightarrow 0$$

A remark on cellular homology

The calculation of the homology of CW-complexes works in all degrees, in particular for negative degrees.
In particular, for any homology theory H and any CW-complex K:

$$H_j(K) = 0 \quad \text{for} \quad j < 0.$$

Here is the reason for this, extracted from the general proof.

* Lecture #6A-ii:

$$\tilde{H}_i(\partial I^n) \cong \tilde{H}_i(S^{n-1}) \cong \begin{cases} G & i = n \\ 0 & i \neq n \end{cases}$$

$$H_i(I^n, \partial I^n) \cong H_i(B^n, S^{n-1}) \cong \begin{cases} G & i = n \\ 0 & i \neq n \end{cases}$$

for $n \geq 0$ and all $i \in \mathbb{Z}$

* Lecture #9B : LES of $(K^{(n)}, K^{(n-1)})$, $j<0$, $n \geq 1$

$$
\begin{array}{ccccccc}
H_{j+1}(K^{(n)}, K^{(n-1)}) & \longrightarrow & H_j(K^{(n-1)}) & \longrightarrow & H_j(K^{(n)}) & \longrightarrow & H_j(K^{(n)}, K^{(n-1)}) \to \cdots \\
\wr\wr & & \| & & \| & & \wr\wr \\
\bigoplus_{6\in I_n} \underbrace{H_{j+1}(I^n, \partial I^n)}_{=0} & \longrightarrow & H_j(K^{(n-1)}) & \longrightarrow & H_j(K^{(n)}) & \longrightarrow & \bigoplus_{6\in I_n} \underbrace{H_j(I^n, \partial I^n)}_{=0}
\end{array}
$$

$\Rightarrow$ $H_j(K^{(n-1)}) \cong H_j(K^{(n)})$ for $j<0$, $n \geq 1$.

* We know $H_j(K^{(0)}) = 0$ for $j<0$ (additivity + normalization axiom).

Hence

$$\forall n \geq 0, \forall j < 0 : H_j(K^{(n)}) = 0 .$$