Split exact sequences

Suppose $0 \longrightarrow A \xrightarrow{i} B \xrightarrow{p} C \longrightarrow 0$

is a split SES, $j: C \longrightarrow B$ a right inverse to p

and $q: B \longrightarrow A$ a left inverse to i.

Then

$$i \oplus j : A \oplus C \longrightarrow B, \quad (a,c) \longmapsto i(a) + j(c)$$

is an isomorphism with inverse

$$q' \oplus p : B \longrightarrow A \oplus C, \quad b \longmapsto (q'(b), p(b))$$

where $q': B \to A$ is the left inverse of i satisfying

$$i \circ q' = \mathrm{id} - j \circ p .$$

Similarly, $i \oplus j' : A \oplus C \to B$ is an isomorphism with inverse $q \oplus p : B \longrightarrow A \oplus C$ where $j': C \to B$ is the unique map with

$$j' \circ p = \mathrm{id} - i \circ q .$$

Proof

* $q': B \to A$ is well-defined: For $b \in B$, we have

$$p(b - j \circ p(b)) = p(b) - p \circ j(p(b)) = 0,$$

thus

$$b - j \circ p(b) \in \ker(p) = \mathrm{im}(i).$$

Since i is injective, there is a unique $a \in A$ with

$$i(a) = b - j \circ p(b).$$

Set $q'(b) := a$. One can check that this defines a homomorphism $q': B \longrightarrow A$.

* q' is a left inverse to i:

$$i \circ q' \circ i = i - j \circ \underbrace{p \circ i}_{=0} = i$$

i is injective, hence $q' \circ i = \mathrm{id}$.

* $(i \oplus j) \circ (q' \oplus p) = i \circ q' + j \circ p = \mathrm{id}$

and

$$(q' \oplus p) \circ (i \oplus j)(a,c) = (q'(i(a) + j(c)), p(i(a) + j(c)))$$
$$= (a + \underbrace{q' \circ j}_{\substack{=0 \\ (*)}}(c), c) = (a,c)$$

((*) follows from

$$i \circ q' \circ j = (\mathrm{id} - j \circ p) \circ j = j - j \circ \underbrace{p \circ j}_{\mathrm{id}} = 0$$

and i being injective.)

$\square$