Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Prof. Paul Biran SS16

Exam in Algebraic Topology I - Summer 2016

Name:

First Name: **Legi-Nr.:**

Please leave the following spaces blank! They will be used by the correctors.

	1. Corr.	2. Corr.	Points	Remarks
Problem 1				
Problem 2				
Problem 3				
Problem 4				
Problem 5				
Problem 6				
Total				
Grade				
Complete?			☐	

Exam in Algebraic Topology I - Summer 2016

Please read carefully!

- The exam is divided into two parts, **Part A** and **Part B**. Part A consists of four problems (1-4) and part B of two problems (5-6). Each problem is divided into sub-problems.
- For **Part A**: Please choose and solve **three out of the four** problems of Part A. **Only three problems will be graded.** You will not get additional points if you solve more than three problems.
- For **Part B**: Please choose and solve only **one out of the two** problems of Part B. **Only one problem will be graded**. You will not get additional points if you solve more than one problem.
- Please only hand in the problems you wish to be graded or very **clearly indicate** which problems you wish to be graded. In Part A only three problems will be graded and in Part B only one problem will be graded.
- In case you hand in too many problems and/or do not clearly indicate which problems you wish to be graded we will only grade the problems that occur first in your work.
- **All answers/statements/counter-examples in your work should be proved.** (It is okay to use theorems/statements proved in class without reproving them.)
- The maximal number of points that can be scored in the exam is 60 and the duration of the exam is 3 hours.
- Do not mix sub-problems from different problems.
- The sub-problems of a problem are not necessarily related to each other.
- Please use a separate sheet of paper for each problem.
- Please do not use red or green pens and do not use pencil.
- Please clearly write your full name on each of the sheets you hand in.
- Please hand in your sheets sorted according to the problem numbers.

Good Luck!

Part A

1. a) [**8 Points**] Let X and Y be topological spaces, and $x_0 \in X$, $y_0 \in Y$ points such that each of them is a deformation retract of some neighbourhood in X and Y respectively. Define
$$X \vee Y = (X \sqcup Y)/x_0 \sim y_0,$$
and denote by $p \in X \vee Y$ the point corresponding to x_0 and y_0. Prove that
$$H_n(X \vee Y, p) \cong H_n(X, x_0) \oplus H_n(Y, y_0)$$
for every n.

   b) [**8 Points**] Recall that the CW-complex $\mathbb{R}P^n$ is obtained from $\mathbb{R}P^{n-1}$ by attaching an n-cell. The infinite union $\mathbb{R}P^\infty = \bigcup_{n\geq 1} \mathbb{R}P^n$ is a CW-complex with one cell in each dimension. It is topologized with the *weak topology*, i.e. a subset $A \subset \mathbb{R}P^\infty$ is closed if and only if $A \cap \mathbb{R}P^n$ is closed in $\mathbb{R}P^n$ for every n.

   Calculate the cellular homology of the CW-complex $\mathbb{R}P^\infty$.

2. a) [**5 Points**] Show that if a map $f : S^n \to S^n$ has no fixed points then $\deg f = (-1)^{n+1}$.

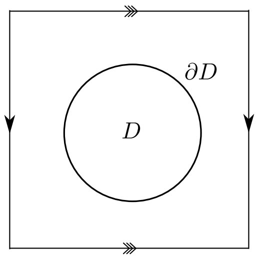
   b) [**5 Points**] Let $g : S^n \to S^n$ be a map of degree 0. Show that there exist points $x, y \in S^n$ such that $g(x) = x$ and $g(y) = -y$.

   c) [**6 Points**] Recall that an *action* of a group G on a space X is a homomorphism $G \to \mathrm{Homeo}(X)$, where $\mathrm{Homeo}(X)$ is the group of homeomorphisms $X \to X$. The action is *free* if the homeomorphism corresponding to each nontrivial element of G has no fixed points.

   Let G be a group acting freely on S^n. Prove that if n is even, then G is either trivial or isomorphic to $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$.

3. a) [**8 Points**] Let $f : \mathbb{R}P^2 \to \mathbb{R}P^2$ be a map which induces an isomorphism on $H_1(\mathbb{R}P^2)$. Prove that f is surjective.

   b) [**8 Points**] Consider the torus T^2 and an embedded disc $D \subset T^2$ as in the picture. Set $X = T^2 \setminus \operatorname{int} D$. Prove that X does not retract onto $\partial D \subset X$.



4. Let X be a space such that $H_i(X) = 0$ for all but finitely many $i \in \mathbb{Z}$ and such that $H_i(X)$ is finitely generated for every i. The *Euler characteristic* of X is defined as

$$\chi(X) := \sum_i (-1)^i \text{rank}\, H_i(X).$$

a) **[9 Points]** Let X be a finite CW-complex and let a_i be the number of i-cells of X. Show that

$$\chi(X) = \sum_i (-1)^i a_i.$$

b) **[7 Points]** Show that if $p : X \to Y$ is a k-sheeted covering map and Y is a finite CW-complex then X is also a finite CW-complex and $\chi(X) = k \cdot \chi(Y)$.

Part B

---

5. Let $f : C_\bullet \to D_\bullet$ be a chain map between the two chain complexes $C_\bullet$ and $D_\bullet$. Recall that the *mapping cone* of f is the chain complex $(cone(f)_\bullet, \mathbf{d})$ defined by:

$$cone(f)_i = C_{i-1} \oplus D_i \quad \text{for all } i \quad \text{and} \quad \mathbf{d}(c, d) = (-\mathrm{d}_C(c), \mathrm{d}_D(d) - f(c))$$

for $c \in C_\bullet$ and $d \in D_\bullet$, where d_C and d_D are the differentials of $C_\bullet$ and $D_\bullet$, respectively.

   a) [**4 Points**] Show that there is a short exact sequence of chain complexes

   $$0 \to D \to cone(f) \to C[-1] \to 0$$

   where $cone(f)_\bullet$ is the mapping cone of f.

   b) [**6 Points**] Show that the connecting homomorphism of the homology long exact sequence induced from the sequence in a) is given by the induced map $f_* : H_*(C) \to H_*(D)$.

   c) [**2 Points**] Show that a chain map $f : C_\bullet \to D_\bullet$ is a quasi-isomorphism if and only if the mapping cone $cone(f)_\bullet$ is an acyclic chain complex.

   Recall: A chain map between two chain complexes is a *quasi-isomorphism* if the induced map on homology is an isomorphism.

6. Compute $\widetilde{H}_i(S^n \setminus X)$ where $X \subsetneq S^n$ and

   a) [**6 Points**] $X \approx S^k \vee S^\ell, \quad 0 < \ell, k < n.$

   b) [**6 Points**] $X \approx S^k \sqcup S^\ell, \quad 0 < \ell, k < n.$