ETH
Eidgenössische Technische Hochschule Zürich
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Exam in Algebraic Topology I - Winter 2016

Name:

First Name: **Legi-Nr.:**

Please leave the following spaces blank! They will be used by the correctors.

	1. Corr.	2. Corr.	Points	Remarks
Problem 1				
Problem 2				
Problem 3				
Problem 4				
Problem 5				
Problem 6				
Total				
Grade				
Complete?			☐	

Exam in Algebraic Topology I - Winter 2016

Please read carefully!

- The exam is divided into two parts, **Part A** and **Part B**. Part A consists of four problems (1-4) and part B of two problems (5-6). Each problem is divided into sub-problems.
- For **Part A**: Please choose and solve **three out of the four** problems of Part A. **Only three problems will be graded.** You will not get additional points if you solve more than three problems.
- For **Part B**: Please choose and solve only **one out of the two** problems of Part B. **Only one problem will be graded**. You will not get additional points if you solve more than one problem.
- Please only hand in the problems you wish to be graded or very **clearly indicate** which problems you wish to be graded. In Part A only three problems will be graded and in Part B only one problem will be graded.
- In case you hand in too many problems and/or do not clearly indicate which problems you wish to be graded we will only grade the problems that occur first in your work.
- **All answers/statements/counter-examples in your work should be proved.** (It is okay to use theorems/statements proved in class without reproving them.)
- The maximal number of points that can be scored in the exam is 60 and the duration of the exam is 3 hours.
- Do not mix sub-problems from different problems.
- The sub-problems of a problem are not necessarily related to each other.
- Please use a separate sheet of paper for each problem.
- Please do not use red or green pens and do not use pencil.
- Please clearly write your full name on each of the sheets you hand in.
- Please hand in your sheets sorted according to the problem numbers.

Good Luck!

Part A

1. Let A be a subspace of a space X and let $r : X \to A$ be a retraction.
   a) [*4* ***Points***] Show that the inclusion $i : A \hookrightarrow X$ induces an injective map on homology.
   b) [*4* ***Points***] Show that $H_j(A) \oplus H_j(X, A) \cong H_j(X)$ for every j.
   c) [***8 Points***] Consider $X = Y = S^1 \subset \mathbb{C}$ and let $f : X \to Y$ be the map given by $f(z) = z^2$. Show that there is no retraction $M_f \to X \times \{1\}$, where
   $$M_f := (Y \sqcup (X \times [0, 1]))/f(z) \sim (z, 0)$$
   is the mapping cylinder of f.

2. a) [***6 Points***] Let $n \geq 1$. Prove that any cycle c that represents a non-trivial class in singular homology $H_n(S^n)$ must cover all of S^n (i.e., the union of the images of all simplices constituting c is all of S^n).
   b) [***10 Points***] Let $f : (D^n, S^{n-1}) \to (D^n, S^{n-1})$ be a map which satisfies $\deg f|_{S^{n-1}} \neq 0$. Prove that f is surjective.

3. Let $p \in \mathbb{N}$, $p > 1$. The space $L_p = B^3/\sim$ is obtained from the closed ball $B^3 \subset \mathbb{R}^3$ by identifying points on its boundary $\partial B^3 = S^2$ as follows:
   Given a point in the closed upper hemisphere, rotate it about the vertical axis by the angle $\frac{2\pi}{p}$ and then reflect it through the equator. See Figure 1.
   a) [***6 Points***] Give a CW-complex structure on L_p.
   b) [***10 Points***] Calculate the cellular homology of L_p.

   Remark: L_p is a special case of the so called *lens spaces*.

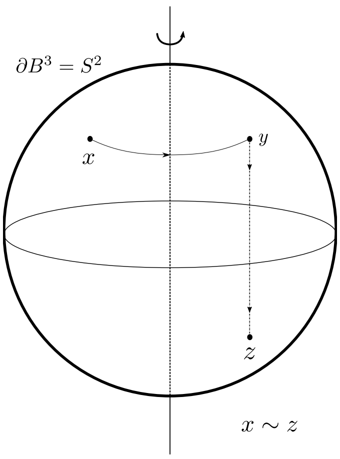

Figure 1: $x \sim z$, where x is any point on the closed upper hemisphere of $\partial B^3 = S^2$, y is the rotation of x about the vertical axis by the angle $\frac{2\pi}{p}$ and z is the reflection of y through the equator.

4. a) [***10 Points***] Let $k > 1$. Show that every map $f : \mathbb{R}P^{2k} \to \mathbb{R}P^{2k}$ has a fixed point.
   b) [***6 Points***] Let $n \geq 1$. Show that for every $k \in \mathbb{Z}$ there exists a map $g_k : S^n \to S^n$ of degree k which has a fixed point.

Part B

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5. Let $C_\bullet$ and $D_\bullet$ be two chain complexes and $\varphi : C_\bullet \to D_\bullet$ a chain map. For each of the following questions, either give a proof or a mathematically justified counter-example.

   a) [*4* ***Points***] Suppose that φ is surjective. Is the induced map φ_* on homology surjective as well?

   b) [*4* ***Points***] Suppose that φ is injective. Is the induced map φ_* on homology injective as well?

   c) [*4* ***Points***] Suppose that φ is bijective. Is the induced map φ_* on homology bijective as well?

6. Let X and Y be finite CW-complexes.

   a) [*4* ***Points***] Let $k \in \mathbb{N}$ and let ω be any $(k-1)$-cell and σ any $(k+1)$-cell of X. Show that $\sum_\tau([\omega : \tau][\tau : \sigma]) = 0$, where τ ranges over all k-cells of X.

      Here $[\omega : \tau]$ stands for the incidence number associated to the cells ω and τ, and similarly for $[\tau : \sigma]$.

   b) [*8* ***Points***] Let $g : X \to Y$ be a cellular map, let η be a k-cell of X and β be a $(k-1)$-cell of Y. Show that

$$\sum_\lambda [\lambda : \eta] \deg(g_{\beta,\lambda}) = \sum_\alpha \deg(g_{\alpha,\eta})[\beta : \alpha].$$

   where λ ranges over all $(k-1)$-cells of X and α ranges over all k-cells of Y.

   Remark: If ρ is an m-cell of X and θ is an m-cell of Y, then the map $g_{\theta,\rho} : S^m \to S^m$ is defined via the diagram:

$$\begin{tikzcd}
B^m \arrow[r, "f_\rho"] \arrow[d, "\gamma_m"'] & X^{(m)} \arrow[r, "g"] \arrow[d] & Y^{(m)} \arrow[r, "p_\theta"] \arrow[d] & S^m \arrow[d, equal] \\
S^m \arrow[r, "\bar{f}_\rho"] \arrow[rrr, bend right, "g_{\theta,\rho}"'] & X^{(m)}/X^{(m-1)} \arrow[r, "\bar{g}"] & Y^{(m)}/Y^{(m-1)} \arrow[r, "\bar{p}_\theta"] & S^m
\end{tikzcd}$$

   where the map f_ρ is the characteristic map (attaching the cell ρ) and p_θ the projection map onto the sphere associated to the cell θ.