Problem set 4

1. Let H be any homology theory. Let X and Y be two non-empty spaces and $f\colon X \to Y$ a map. Show that the homomorphism $f_*\colon H_0(X) \to H_0(Y)$ induces a homomorphism $\tilde{f}_*\colon \widetilde{H}_0(X) \to \widetilde{H}_0(Y)$.

2. For $1 \leq i \leq n$ consider the maps
$$\begin{aligned} \tau_i\colon S^n &\longrightarrow S^n \\ (x_1, \dots, x_{n+1}) &\longmapsto (x_1, \dots, -x_i, \dots, x_{n+1}). \end{aligned}$$
Show that any two of these maps are homotopic.

3. For a space X, denote by $\hat{X}$ its 1-point compactification. Show that if $f\colon X \to Y$ is a homeomorphism then f induces a homeomorphism $\hat{f}\colon \hat{X} \to \hat{Y}$ with $\hat{f}(\infty) = \infty$ and $\hat{f}|_X = f$. What happens if we drop the assumption that f is a homeomorphism?

4. Show that the stereographic projection $\hat{\pi}\colon S^n \to \hat{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$ is a homeomorphism. Derive a formula for $\hat{\pi}$.

5. Construct a nowhere vanishing vector field on the odd-dimensional sphere S^{2k-1}.
Hint: View S^{2k-1} as the unit sphere inside $\mathbb{C}^k$, with respect to the standard Euclidean metric on $\mathbb{C}^k$. For every point $z \in S^{2k-1}$, viewed as a k-tuple of complex numbers, consider the curve $\gamma_z\colon (-\epsilon, \epsilon) \to S^{2k-1}$ given by $\gamma_z(t) = e^{it}z$.

6. (a) If $f : S^n \to S^n$ has no fixed points, then $\deg f = (-1)^{n+1}$.
   (b) Let $f : S^n \to S^n$ be a map of degree 0. Show that there exist points $x, y \in S^n$ such that $f(x) = x$ and $f(y) = -y$.

7. We view S^1 as the unit circle in $\mathbb{C}$. Show that for each $k \in \mathbb{Z}$ the map $f : S^1 \to S^1$, given by $f(z) = z^k$, has degree k.

8. (a) We denote by SX the (unreduced) suspension of a space X we introduced in Exercise 4 in Problem set 3. If $X = S^n$, then $SX \approx S^{n+1}$. Let $Sf : S^{n+1} \to S^{n+1}$ be the suspension of a map $f : S^n \to S^n$. Show that $\deg Sf = \deg f$.
   (b) Show that for each $n \geq 1$ and each $k \in \mathbb{Z}$ there is a map $f : S^n \to S^n$ of degree k.

9. Construct a surjective map $S^n \to S^n$ of degree 0 for each $n \geq 1$.
Hint: Do it first for $n = 1$ and then use exercise 8(a).

10. Let G be a group acting freely on S^n. Prove that if n is even, then G is either trivial or isomorphic to $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$.

11. Consider $SO(n)$ as a subset of $\mathbb{R}^{n^2}$ and endow $SO(n)$ with the induced topology. Show that $SO(n)$ is path connected. Similarly, show that the Lie groups $GL(n, \mathbb{C})$, $GL^+(n, \mathbb{R})$, $U(n)$, $SU(n)$ are path connected.