

Solutions to problem set 3

1. By the exactness axiom, there is a long exact sequence for the pair (X, X) :

$$\dots \rightarrow H_i(X) \xrightarrow{id_* = id} H_i(X) \rightarrow H_i(X, X) \rightarrow H_{i-1}(X) \xrightarrow{id_* = id} H_{i-1}(X) \dots$$

Since id is an isomorphism, it follows from exactness that $H_i(X, X) = \{0\}$. The exactness axiom for the pair (X, \emptyset) for some space X reads

$$\dots \rightarrow H_i(X) \xrightarrow{id_* = id} H_i(X) \cong H_i(X, \emptyset) \rightarrow H_i(\emptyset) \rightarrow H_i(X) \xrightarrow{id_* = id} H_i(X) \cong H_i(X, \emptyset) \rightarrow \dots$$

Again it follows from exactness that $H_i(\emptyset) = 0$ for all i .

2. Let $A = \emptyset$ but $X \neq \emptyset$. It's enough to consider the reduced sequence in degrees ≤ 1 . Consider the diagram

$$\begin{array}{ccccccc}
 & & & 0 & & & \\
 & & & \uparrow & & & \\
 & & & \mathbb{Z} & & & \\
 & & & \uparrow \epsilon_* & & & \\
 H_1(X, \emptyset) & \longrightarrow & H_0(\emptyset) = 0 & \longrightarrow & H_0(X) & \xrightarrow{id} & H_0(X, \emptyset) = H_0(X) \longrightarrow 0 \\
 \parallel & & \parallel & & \uparrow & & \uparrow id \\
 H_1(X, \emptyset) & \longrightarrow & \tilde{H}_0(\emptyset) = 0 & \longrightarrow & \tilde{H}_0(X) & \longrightarrow & H_0(X, \emptyset) = H_0(X) \xrightarrow{\epsilon_*} \tilde{H}_{-1}(\emptyset) = \mathbb{Z} \longrightarrow 0 \\
 & & & & \uparrow & & \\
 & & & & 0 & &
 \end{array}$$

The first row comes from the (non-reduced) LES for (X, \emptyset) . The lower row is the sequence of reduced homologies, for which we need to show exactness. The SES occurring as a column comes from the definition of reduced homology. By comparing this SES with the lower row it follows that the reduced sequence is also exact.

Assume now that $X = \emptyset$. Then the sequence for reduced homology is 0 except in degree -1 :

$$H_0(X, A) = 0 \rightarrow H_{-1}(A) = \mathbb{Z} \xrightarrow{id_* = id} \mathbb{Z} = H_{-1}(X) \rightarrow 0 = H_{-1}(X, A).$$

This is exact.

3. Consider the reduced long exact sequence for the pair (X, A) :

$$\dots \rightarrow \tilde{H}_p(A) \rightarrow \tilde{H}_p(X) \rightarrow H_p(X, A) \rightarrow \tilde{H}_{p-1}(A) \rightarrow \dots$$

Since $\tilde{H}_*(A) = 0$ by assumption, this long exact sequence yields exact sequences

$$0 \rightarrow \tilde{H}_p(X) \rightarrow H_p(X, A) \rightarrow 0$$

which tell us that

$$\tilde{H}_p(X) \cong H_p(X, A)$$

for all p .

4. Let U be a small open neighbourhood of the "bottom" point $[0, x] \in \Sigma X$. We apply excision to the pair $(\Sigma X, C_- X)$ and the open subset U . We get an isomorphism

$$H_i(\Sigma X \setminus U, C_- X \setminus U) \xrightarrow{\text{incl}_*} H_i(\Sigma X, C_- X).$$

Note that $\Sigma X \setminus U$ deformation retracts to $C_+ X$ relative to X . By the homotopy axiom, the inclusion of $(C_+ X, X)$ into $(\Sigma X \setminus U, C_- X \setminus U)$ induces an isomorphism in homology:

$$H_i(C_+ X, X) \xrightarrow{\text{incl}_*} H_i(\Sigma X \setminus U, C_- X \setminus U).$$

Put together, we get isomorphisms:

$$\tilde{H}_{i-1}(X) \xleftarrow{\partial_*} H_i(C_+ X, X) \xrightarrow{\text{incl}_*} H_i(\Sigma X \setminus U, C_- X \setminus U) \xrightarrow{\text{incl}_*} H_i(\Sigma X, C_- X) \xleftarrow{\text{incl}_*} \tilde{H}_i(\Sigma X),$$

where the first isomorphism comes from the LES of the pair $(C_+ X, X)$ and the last isomorphism comes from the LES of the pair $(\Sigma X, C_- X)$. (One uses that the cones $C_{\pm} X$ are contractible and hence acyclic.) Each of the isomorphism is natural, and hence the composition

$$\tilde{H}_{i-1}(X) \xrightarrow{\cong} \tilde{H}_i(\Sigma X)$$

is a natural isomorphism.

5. Consider the LES for the pair $(X \vee Y, X)$:

$$\cdots \rightarrow H_{n+1}(X \vee Y, X) \xrightarrow{\partial_*} \tilde{H}_n(X) \xrightarrow{(i_X)_*} \tilde{H}_n(X \vee Y) \xrightarrow{k_*} H_n(X \vee Y, X) \rightarrow \cdots \quad (1)$$

We claim that this long exact sequence splits into many short exact sequences. Denote by $i_Y: Y \rightarrow X \vee Y$ and $j: (Y, \emptyset) \rightarrow (X \vee Y, X)$ the inclusions. Then j induces an isomorphism in homology:

$$j_* = \left(\tilde{H}_n(Y) \xrightarrow{\text{incl}_* \cong} H_n(Y, \{y_0\}) \xrightarrow{\text{incl}_* \cong} H_n(Y \cup N, N) \xrightarrow{\text{incl}_* \cong} H_n(X \vee Y, X) \right). \quad (2)$$

Indeed, the first isomorphism in (2) comes from the LES for the pair $(X, \{y_0\})$. The second isomorphism in (2) follows from the homotopy axiom because $Y \cup N$ deformation retracts to Y relative to $\{y_0\}$. Finally, the third isomorphism in (2) comes from excision for the pair $(X \vee Y, X)$ with respect to the open set $(X \vee Y) \setminus (Y \cup N)$.

Note that $k_* \circ (i_Y)_* \circ (j_*)^{-1} = id$. This implies that k_* is surjective and the LES (1) splits into short exact sequences:

$$0 \rightarrow \tilde{H}_n(X) \xrightarrow{\text{incl}_*} \tilde{H}_n(X \vee Y) \xrightarrow{k_*} H_n(X \vee Y, X) \rightarrow 0.$$

Replacing $H_n(X \vee Y, X)$ with $\tilde{H}_n(Y)$ via j_* , we get SES's

$$0 \rightarrow \tilde{H}_n(X) \xrightarrow{\text{incl}_*} \tilde{H}_n(X \vee Y) \xrightarrow{(p_Y)_*} \tilde{H}_n(Y) \rightarrow 0,$$

where $p_Y: X \vee Y \rightarrow Y$ denotes the projection. This SES's split with a right splitting given by $(i_Y)_*$ and a left splitting given by $(p_X)_*$. Therefore, $(i_X)_* \oplus (i_Y)_*: H_n(X) \oplus H_n(Y) \rightarrow H_n(X \vee Y)$ is an isomorphism with inverse $p_X \oplus p_Y$.

6. The commutativity of the diagram

$$\begin{array}{ccc}
 (Y, \emptyset) & \xrightarrow{k} & (X \sqcup Y, X) \\
 & \searrow i_Y & \nearrow j \\
 & X \sqcup Y &
 \end{array}$$

implies that

$$k_* = j_* \circ (i_Y)_* : H_*(Y) \rightarrow H_*(X \sqcup Y, X) \quad (3)$$

by functoriality of H . Since k_* is an isomorphism by the Excision axiom, it follows that j_* is surjective. This implies that the connecting homomorphisms ∂_* in the long exact sequence for the pair $(X \sqcup Y, X)$ are all zero, and hence this long exact sequence gives rise to short exact sequences

$$0 \rightarrow H_p(X) \xrightarrow{(i_X)_*} H_p(X \sqcup Y) \xrightarrow{j_*} H_p(X \sqcup Y, X) \rightarrow 0. \quad (4)$$

Now equation (3) is equivalent to $j_* \circ (i_Y)_* \circ k_*^{-1} = \text{id}_{H_p(X \sqcup Y, X)}$, and thus j_* has a right inverse. Hence the short exact sequence (4) splits, and therefore

$$(i_X)_* \oplus ((i_Y)_* \circ k_*^{-1}) : H_p(X) \oplus H_p(X \sqcup Y, X) \rightarrow H_p(X \sqcup Y)$$

is an isomorphism; precomposing it with the isomorphism $\text{id}_{H_p(X)} \oplus k_*$ yields the desired isomorphism

$$(i_X)_* \oplus (i_Y)_* : H_p(X) \oplus H_p(Y) \rightarrow H_p(X \sqcup Y).$$