Solutions to problem set 4

1. Consider the commutative diagram

$$\begin{tikzcd} H_0(X) \arrow[r, "(\epsilon_X)*"] \arrow[d, "f_*"] & H_0(P) \arrow[d, "id"] \\ H_0(Y) \arrow[r, "(\epsilon_Y)_*"] & H_0(P). \end{tikzcd}$$

Here P denotes the one-point space and $(\epsilon_X)*$ is the homomorphism induced by the unique map $\epsilon_X : X \to P$. Similarly for ϵ_Y. It follows from a simple diagram chase that f_* induces a homomorphism

$$f_* : \widetilde{H}_0(X) = \ker((\epsilon_X)*) \to \ker((\epsilon_Y)_*) = \widetilde{H}_0(Y).$$

2. Let $1 \leq i \neq j \leq n$. Consider the following rotations $R_t^{i,j}$ in $\mathbb{R}^{n+1}$: In the x_i-x_j-plane, $R_t^{i,j}$ is represented by the matrix

$$R_t^{i,j} = \begin{pmatrix} \cos(\frac{\pi}{2}t) & \sin(\frac{\pi}{2}t) \\ -\sin(\frac{\pi}{2}t) & \cos(\frac{\pi}{2}t) \end{pmatrix}.$$

$R_t^{i,j}$ fixes the other coordinates x_k, $k \neq i, j$. Then $R_t^{i,j}$ restrict to homeomorphisms on S^n and $\tau_i = (R_1^{i,j})^{-1} \circ \tau_j \circ R_1^{i,j}$. Thus $\left\{(R_t^{i,j})^{-1} \circ \tau_j \circ R_t^{i,j}\right\}_{t\in[0,1]}$ is a homotopy from τ_j to τ_i.

3. We show that $\hat{f}$ is continuous: Let $V \subset \hat{Y}$ be an open subset. If $V \subset Y$ is open, then $\hat{f}^{-1}(V) = f^{-1}(V) \subset X$ is open in X by continuity of f. Thus $\hat{f}^{-1}(V)$ is open in $\hat{X}$. If $\infty \in V$, then $\infty \in \hat{f}^{-1}(V)$ and $\hat{X} \backslash \hat{f}^{-1}(V) = f^{-1}(\hat{Y} \backslash V)$. Note that $\hat{Y} \backslash V \subset Y$ is compact. Since f is a homeomorphism, f is proper and so $f^{-1}(\hat{Y} \backslash V)$ is compact. It follows that $\hat{f}^{-1}(V)$ is open in $\hat{X}$. The same argument applied to f^{-1} implies that $\widehat{f^{-1}}$ is continuous. Thus $\hat{f}$ is a homeomorphism with inverse $\widehat{f^{-1}}$.

   Dropping the assumption that f is a homeomorphism is not possible: Consider the inclusion i of the 1−disk $B_1(0)$ into the 2−disk $B_2(0)$. Then there is no continuous extension of i to the compactifications. (The complement of $\overline{B_1(0)} \subset \widehat{B_2(0)}$ is an open neighbourhood of ∞ and its inverse image $\{\infty\} \in \widehat{B_1(0)}$ is not open.)

   Proper continuous maps can be extended to continuous maps on the compactifications. See also Bredon, Theorem 11.4.

4. View S^n as the standard sphere in $\mathbb{R}^{n+1}$ with coordinates $(x_0, x_1, \ldots, x_n)$. Define the stereographic projection $\pi : S^n \backslash \{(1, 0, \ldots, 0)\} \to \mathbb{R}^n$ as follows:

$$\pi(x_0, x_1, \ldots, x_n) = \left(\frac{x_1}{1 - x_0}, \ldots, \frac{x_n}{1 - x_0}\right).$$

$\pi(x)$ is the intersection of the unique line trough x and $(1, 0 \ldots, 0)$ with the hyperplane $\{x_0 = 0\}$. π is a homeomorphism with inverse

$$\pi^{-1}(y_1, \ldots, y_n) = \left(\frac{||y||^2 - 1}{||y||^2 + 1}, \frac{2y_1}{||y||^2 + 1}, \ldots, \frac{2y_n}{||y||^2 + 1}\right).$$

It follows from Exercise 3 that π extends to a homeomorphism $\widehat{\pi} : S^n \to \mathbb{R}^n \cup \{\infty\}$.

5. View S^{2k-1} as the unit sphere inside $\mathbb{C}^k$, with respect to the standard Euclidean metric on $\mathbb{C}^k$. For every point $z \in S^{2k-1}$, viewed as a k-tuple of complex numbers, consider the curve $\gamma_z \colon (-\epsilon, \epsilon) \to S^{2k-1}$ given by $\gamma_z(t) = e^{it}z$. Consider the vector field X on S^{2k-1} given by

$$X(z) = \dot{\gamma}_z(0).$$

This is a smooth nonwhere vanishing vector field. In real coordinates it is given by

$$X(x_1, y_1, \ldots, x_k, y_k) = (-y_1, x_1, \ldots, -y_k, x_k).$$

6. (a) Since $f(x) \neq x$, $\forall x \in S^n$, the line segment $(1-t)f(x) - tx$, $t \in [0,1]$, does not pass through 0. Therefore, if f has no fixed points,

$$f_t(x) := \frac{(1-t)f(x) - tx}{|(1-t)f(x) - tx|}$$

is a well defined homotopy from f to the antipodal map $-id$ which has degree $\deg(-id) = (-1)^{n+1}$. Thus $\deg f = (-1)^{n+1}$.

(b) Since $\deg f = 0 \neq (-1)^{n+1}$ it must have a fixed point $x \in S^n$ by exercise 6.(a), i.e. $f(x) = x$. Similarly, since $g := (-id) \circ f$ has degree $\deg g = \deg(-id) \cdot \deg f = 0$, there is a fixed point $y \in S^n$ of g, i.e. $g(y) = -f(y) = y$. This means that $f(y) = -y$.

7. See example 2.32 on page 137 in Hatcher's book.

8. (a) Recall from Exercise 4 in Problem set 4 that we have the following commutative diagram

$$\begin{tikzcd} \widetilde{H}_{n+1}(S^{n+1}) \arrow[r, "\partial_*", "\cong"'] \arrow[d, "(Sf)_*"] & \widetilde{H}_n(S^n) \arrow[d, "f_*"] \\ \widetilde{H}_{n+1}(S^{n+1}) \arrow[r, "\partial_*", "\cong"'] & \widetilde{H}_n(S^n) \end{tikzcd}$$

Therefore, if f_* is multiplication by $d = \deg f$, then $(Sf)_*$ is also multiplication by d and hence $\deg f = \deg Sf$.

(b) Given $k \in \mathbb{Z}$ the map $S^1 \to S^1 : z \mapsto z^k$ has degree k. Now assume that we have constructed a map $f : S^n \to S^n$ of degree k, then (by exercise 8.(a)), the map $Sf : S^{n+1} \to S^{n+1}$ has degree k as well. So the claim follows by induction.

9. First, let $n = 1$ and denote $I := [0,1]$. Let $g : I \to \mathbb{R}$ be a continuous map such that $g(0) = g(1) = 0$ and $g(1/2) = 2\pi$. The map g induces a well defined continuous surjection $f : I/\partial I = S^1 \to S^1 : t \mapsto e^{ig(t)}$. By the path lifting property the map g is the unique lift of f to the universal cover $\mathbb{R}$ of S^1 starting at the point $0 \in \mathbb{R}$. So $f \in p_\sharp \underbrace{\pi_1(\mathbb{R}, 0)}_{=0} \subset \pi_1(S^1, 1)$ is homotopic to a constant map (which is clearly not surjective and therefore has degree 0) and hence $\deg f = 0$. Here, $p : \mathbb{R} \to S^1$ is the universal cover.

Using exercise 8.(a) we obtain, by repeatedly suspending the map f, a surjective map $S^n \to S^n$ of degree 0.

For an alternative, more explicit, solution see example 2.31 in Hatcher's book.

10. Let the group action be given by the homomorphism $\rho : G \to Homeo(S^n)$. The degree of a homoemorphism is always ± 1. Therefore the group action determines a degree function $d : G \to \{\pm 1\}$ given by $d(g) := \deg \rho(g)$. Furthermore d is a homomorhism:

$$d(hg) = \deg \rho(hg) = \deg(\rho(h) \circ \rho(g)) = \deg \rho(h) \cdot \deg \rho(g) = d(h) \cdot d(g).$$

If $g \in G$ is a non trivial element, then $\rho(g)$ has no fixed points as the action is free and hence (by exercise 6.(a)) we have $d(g) = (-1)^{n+1}$. So, if n is even, the kernel of d is trivial which implies that G is isomorphic to a subgroup of $\{\pm 1\} \cong \mathbb{Z}_2$.

11. For $n = 2$ we have that $SO(2)$ is homeomorphic to the circle S^1 which is path connected. Proceeding by induction we assume that $SO(n-1)$ is path connected. Given any $A \in SO(n)$ it is enough to show that there is a path in $SO(n)$ connecting A to the identity matrix I_n. This means that we need to find a continuous path taking the standard basis $e_1, \ldots, e_n$ to their image $Ae_1, \ldots, Ae_n$. Let $\Lambda \subset \mathbb{R}^n$ be a plane containig both e_1 and Ae_1. By the path connectedness of $SO(2)$, we can continuously move e_1 to Ae_1 by a rotation R of the plane Λ.

    It remains to continuously move $Re_2, \ldots, Re_n$ to $Ae_2, \ldots, Ae_n$ while keeping Ae_1 fixed. Notice that $Ae_1 = Re_1 \perp Re_i$ and $Ae_1 \perp Ae_i$ for each $2 \leq i \leq n$ since both R and A preserve angles. Hence the required motion can take place in the hyperplane $\mathbb{R}^{n-1}$ of vectors orthogonal to Ae_1, where it exists by the assumption that $SO(n-1)$ is path connected.

    Concatenating the two motions gives a path in $SO(n)$ from I_n to A and thus $SO(n)$ is path connected.

    For the other groups, take a look at
    `https://www.jnu.ac.in/Faculty/vedgupta/matrix-gps-gupta-mishra.pdf`