Prof. Dr. A. Iozzi | Introduction to Lie Groups | Autumn 2020
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Exercise Sheet 1

Exercise 1.(Unitary Operators):

Let $\mathcal{H}$ be a Hilbert space and $U(\mathcal{H})$ its group of unitary operators. Show that the weak operator topology coincides with the strong operator topology on $U(\mathcal{H})$.

Exercise 2.(Compact-Open Topology):

Let X, Y, Z be topological space, and denote by $C(Y,X) := \{f\colon Y \to X \text{ continuous}\}$ the set of continuous maps from Y to X. The set $C(Y,X)$ can be endowed with the *compact-open topology*, that is generated by the subbasic sets

$$S(K,U) := \{f \in C(Y,X) \,|\, f(K) \subseteq U\},$$

where $K \subseteq Y$ is compact and $U \subseteq X$ is open.

Prove the following useful facts about the compact-open topology.
If Y is locally compact, then:

a) The evaluation map $e\colon C(Y,X) \times Y \to X, e(f,y) := f(y)$, is continuous.

b) A map $f\colon Y \times Z \to X$ is continuous if and only if the map

$$\hat{f}\colon Z \to C(Y,X), \hat{f}(z)(y) = f(y,z),$$

is continuous.

Exercise 3.(General Linear Group $\mathrm{GL}(n,\mathbb{R})$):

The general linear group

$$\mathrm{GL}(n,\mathbb{R}) := \{A \in \mathbb{R}^{n\times n} \,|\, \det A \neq 0\} \subseteq \mathbb{R}^{n\times n}$$

is naturally endowed with the subspace topology of $\mathbb{R}^{n\times n} \cong \mathbb{R}^{n^2}$. However, it can also be seen as a subset of the space of homeomorphisms of $\mathbb{R}^n$ via the injection

$$j\colon \mathrm{GL}(n,\mathbb{R}) \to \mathrm{Homeo}(\mathbb{R}^n),$$
$$A \mapsto (x \mapsto Ax).$$

a) Show that $j(\mathrm{GL}(n,\mathbb{R})) \subset \mathrm{Homeo}(\mathbb{R}^n)$ is a closed subset, where $\mathrm{Homeo}(\mathbb{R}^n) \subset$

$C(\mathbb{R}^n, \mathbb{R}^n)$ is endowed with the compact-open topology.

b) If we identify $\mathrm{GL}(n, \mathbb{R})$ with its image $j(\mathrm{GL}(n, \mathbb{R})) \subset \mathrm{Homeo}(\mathbb{R}^n)$ we can endow it with the induced subspace topology. Show that this topology coincides with the usual topology coming from the inclusion $\mathrm{GL}(n, \mathbb{R}) \subset \mathbb{R}^{n \times n}$.

Hint: Exercise 2 can be useful here.

Exercise 4.(Isometry Group $\mathrm{Iso}(X)$):

Let (X, d) be a *compact* metric space. Recall that the isometry group of X is defined as

$$\mathrm{Iso}(X) = \{f \in \mathrm{Homeo}(X) : d(f(x), f(y)) = d(x, y) \quad \text{for all } x, y \in X\}.$$

Show that $\mathrm{Iso}(X) \subset \mathrm{Homeo}(X)$ is compact with respect to the compact-open topology.

Hint: Use the fact that the compact-open topology is induced by the metric of uniform-convergence and apply Arzelà–Ascoli's theorem.

Exercise 5.(p-adic Integers $\mathbb{Z}_p$):

Let $p \in \mathbb{N}$ be a prime number. Recall that the p-adic integers $\mathbb{Z}_p$ can be seen as the subspace

$$\left\{(a_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} \mathbb{Z}/p^n\mathbb{Z} : a_{n+1} \equiv a_n \,(\mathrm{mod}\ p^n)\right\}$$

of the infinite product $\prod_{n\in\mathbb{N}} \mathbb{Z}/p^n\mathbb{Z}_p$ carrying the induced topology. Note that each $\mathbb{Z}/p^n\mathbb{Z}$ carries the discrete topology and $\prod_{n\in\mathbb{N}} \mathbb{Z}/p^n\mathbb{Z}$ is endowed with the resulting product topology.

a) Show that the image of $\mathbb{Z}$ via the embedding

$$\begin{aligned} \iota : \mathbb{Z} &\to \mathbb{Z}_p, \\ x &\mapsto (x \,(\mathrm{mod}\ p^n))_{n\in\mathbb{N}} \end{aligned}$$

is dense. In particular, $\mathbb{Z}_p$ is a compactification of $\mathbb{Z}$.

b) Show that the 2-adic integers $\mathbb{Z}_2$ are homeomorphic to the "middle thirds" cantor set C as defined in Exercise 6b).

Exercise 6†.(Homeomorphism Group $\mathrm{Homeo}(X)$):

a) Let X be a *compact* Hausdorff space. Show that $(\mathrm{Homeo}(X), \circ)$ is a topological group when endowed with the compact-open topology.

b) The objective of this exercise is to show that $(\mathrm{Homeo}(X), \circ)$ will not necessarily be a topological group if X is only locally compact.

Consider the "middle thirds" Cantor set

$$C = \left\{ \sum_{n=1}^{\infty} \varepsilon_n 3^{-n} : \varepsilon_n \in \{0,2\} \text{ for each } n \in \mathbb{N} \right\} \subset [0,1]$$

in the unit interval. We define the sets $U_n = C \cap [0, 3^{-n}]$ and $V_n = C \cap [1 - 3^{-n}, 1]$. Further we construct a sequence of homeomorphisms $h_n \in \mathrm{Homeo}(C)$ as follows:

- $h_n(x) = x$ for all $x \in C \setminus (U_n \cup V_n)$,
- $h_n(0) = 0$,
- $h_n(U_{n+1}) = U_n$,
- $h_n(U_n \setminus U_{n+1}) = V_{n+1}$,
- $h_n(V_n) = V_n \setminus V_{n+1}$.

These restrict to homeomorphisms $h_n|_X$ on $X := C \setminus \{0\}$.

Show that the sequence $(h_n|_X)_{n \in \mathbb{N}} \subset \mathrm{Homeo}(X)$ converges to the identity on X but the sequence $((h_n|_X)^{-1})_{n \in \mathbb{N}} \subset \mathrm{Homeo}(X)$ of their inverses does not!

Remark: However, if X is locally compact and *locally connected* then $\mathrm{Homeo}(X)$ is a topological group.

c) Let $\mathbb{S}^1 \subset \mathbb{C} \setminus \{0\}$ denote the circle. Show that $\mathrm{Homeo}(\mathbb{S}^1)$ is not locally compact.

Remark: In fact, $\mathrm{Homeo}(M)$ is not locally compact for any manifold M.

Due date: Thursday, 01/10/2020

Please, upload your solution via the SAM upload tool.

In order to access the website you will need a NETHZ-account and you will have to be connected to the ETH-network. From outside the ETH network you can connect to the ETH network via VPN. Here are instructions on how to do that.

Make sure that your solution is **one PDF file** and that its **file name** is formatted in the following way:

`solution_<number exercise sheet>_<last name>_<first name>.pdf`

For example: If your first name is Alice, your last name is Miller, and you want to hand-in your solution to Exercise Sheet 2, then you will have to upload your solution as **one** PDF file with the following file name:

`solution_2_Miller_Alice.pdf`

Solutions that fail to comply with the above requirements will be ignored.