Prof. Dr. A. Iozzi | Introduction to Lie Groups | Autumn 2020
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Exercise Sheet 3

Exercise 1.(Regular Subgroups are closed):

Let G be a Lie group, $H \leq G$ a subgroup that is also a regular submanifold. Prove that H is a closed subgroup of G.

Exercise 2.(Non-closed Subgroup):

Give an example of a Lie group G and a subgroup $H < G$ that is not closed and not a Lie group with the topology induced from G.

Exercise 3.(Differential of det**):**

We consider the determinant function $\det : \mathrm{GL}(n, \mathbb{R}) \to \mathbb{R}^*$. Show that its differential at the identity matrix I is the trace function

$$D_I \det = \mathrm{tr}.$$

Exercise 4.(Dimension of O$(n, \mathbb{R})$):

Show that the dimension of $\mathrm{O}(n, \mathbb{R})$ is $n(n-1)/2$.

Exercise 5.(The Matrix Lie Groups $O(p,q)$ and $U(p,q)$):

Let $p, q \in \mathbb{N}$ and $n = p + q$.

a) We define the (indefinite) symmetric bilinear form $\langle \cdot, \cdot \rangle_{p,q}$ of signature (p,q) on $\mathbb{R}^n$ to be

$$\langle v, w \rangle_{p,q} := v_1 w_1 + \cdots + v_p w_p - v_{p+1} w_{p+1} - \cdots - v_{p+q} w_{p+q}$$

for all $v = (v_1, \ldots, v_n), w = (w_1, \ldots, w_n) \in \mathbb{R}^n$. As the orthogonal group $O(n)$ is defined to be the group of matrices that preserve the standard Euclidean inner product we may now define $O(p,q)$ to be the group of matrices that preserve the above bilinear form:

$$O(p,q) := \left\{ A \in \mathrm{GL}(n, \mathbb{R}) : \langle Av, Aw \rangle_{p,q} = \langle v, w \rangle_{p,q} \quad \forall v, w \in \mathbb{R}^n \right\}.$$

Show that $O(p,q)$ is a Lie group using the inverse function theorem/constant rank theorem. What is its dimension?

b) Similarly we may define the following symmetric sesquilinear form on $\mathbb{C}^n$

$$\langle w,z\rangle_{p,q} := \bar{w}_1 z_1 + \cdots + \bar{w}_p z_p - \bar{w}_{p+1} z_{p+1} - \cdots - \bar{w}_{p+q} z_{p+q}$$

for all $w = (w_1,\ldots,w_n), z = (z_1,\ldots,z_n) \in \mathbb{C}^n$, and

$$U(p,q) = \{A \in \mathrm{GL}(n,\mathbb{C}) : \langle Aw, Az\rangle_{p,q} = \langle w,z\rangle_{p,q} \quad \forall w,z \in \mathbb{C}^n\}.$$

Show that $U(p,q)$ is a (real) Lie group using the inverse function theorem/constant rank theorem. What is its (real) dimension?

Exercise 6.(One- and two-dimensional Lie Algebras):

Classify the one- and two-dimensional real Lie algebras up to Lie algebra isomorphism and realize them as Lie subalgebras of some $\mathfrak{gl}_n\mathbb{R} = \mathfrak{gl}(\mathbb{R}^n)$.

Hint: In dimension two one can show that if the Lie algebra is non-abelian then there is a basis X, Y such that $[X,Y] = Y$.

Exercise 7.(The adjoint representation ad):

Let V be a vector space over a field k.

a) Show that the vector space of endomorphisms

$$\mathfrak{gl}(V) := \{A\colon V \to V \text{ linear}\}$$

is a Lie algebra with the Lie bracket given by the commutator

$$[A,B] := AB - BA$$

for all $A, B \in \mathfrak{gl}(V)$.

b) Let $\mathfrak{g}$ be a Lie algebra over k. The *adjoint representation*

$$\mathrm{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$$

is defined as $\mathrm{ad}(X)(Y) := [X,Y]$ for all $X, Y \in \mathfrak{g}$. Show that ad is a Lie algebra homomorphism.

Exercise 8.(Quaternions):

Let $\mathbb{H} := \mathbb{R} + \mathbb{R}\mathbf{i} + \mathbb{R}\mathbf{j} + \mathbb{R}\mathbf{k}$ and define – in addition to the $\mathbb{R}$ vector space structure – a multiplication on $\mathbb{H}$ by requiring:

$$\begin{aligned}\mathbf{ij} &= \mathbf{k} = -\mathbf{ji},\\ \mathbf{jk} &= \mathbf{i} = -\mathbf{kj},\\ \mathbf{ki} &= \mathbf{j} = -\mathbf{ik},\\ \mathbf{i}^2 &= \mathbf{j}^2 = \mathbf{k}^2 = -1.\end{aligned}$$

The resulting skew-field is called the Hamiltonian quaternions.

a) Prove that there is a ring isomorphism:

$$\mathbb{H} \cong \left\{ \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \middle| a, b \in \mathbb{C} \right\}.$$

b) Define a Lie bracket on $\mathbb{H}$ by $[u,v] := uv - vu$. Show that $V = \mathbb{R}\mathbf{i} + \mathbb{R}\mathbf{j} + \mathbb{R}\mathbf{k}$ is a Lie ideal in $\mathbb{H}$ and that the Lie subalgebra $(V, [\cdot,\cdot])$ is isomorphic to the Lie algebra $\mathbb{R}^3$ with the cross product

$$x \times y = (x_2y_3 - y_2x_3, x_3y_1 - y_3x_1, x_1y_2 - y_1x_2) \quad \forall x, y \in \mathbb{R}^3$$

as a Lie bracket.

<u>Remark:</u> A *Lie ideal* in a Lie algebra $\mathfrak{g}$ is a Lie subalgebra $\mathfrak{i} \subseteq \mathfrak{g}$ such that $[X,Y] \in \mathfrak{i}$ for all $X \in \mathfrak{g}, Y \in \mathfrak{i}$.

Due date: Thursday, 29/10/2020

Please, upload your solution via the SAM upload tool.

In order to access the website you will need a NETHZ-account and you will have to be connected to the ETH-network. From outside the ETH network you can connect to the ETH network via VPN. Here are instructions on how to do that.

Make sure that your solution is **one PDF file** and that its **file name** is formatted in the following way:

`solution_<number exercise sheet>_<last name>_<first name>.pdf`

For example: If your first name is Alice, your last name is Miller, and you want to hand-in your solution to Exercise Sheet 2, then you will have to upload your solution as **one** PDF file with the following file name:

`solution_2_Miller_Alice.pdf`

Solutions that fail to comply with the above requirements will be ignored.