Exercise Sheet 4

Exercise 1.(Related Vector Fields):

Let M, N be smooth manifolds and let $\varphi : M \to N$ be a smooth map. Recall that two vector fields $X \in \mathrm{Vect}(M)$, $X' \in \mathrm{Vect}(N)$ are called φ-related if

$$d_p\varphi(X_p) = X'_{\varphi(p)}$$

for every $p \in M$.

Show that $[X,Y]$ is φ-related to $[X',Y']$ if $X \in \mathrm{Vect}(M)$ is φ-related to $X' \in \mathrm{Vect}(N)$ and $Y \in \mathrm{Vect}(M)$ is φ-related to $Y' \in \mathrm{Vect}(N)$.

Exercise 2.(Leibniz Rule):

Let $A, B : (-\varepsilon, \varepsilon) \to \mathbb{R}^{n\times n}$ be smooth curves and define $\varphi : (-\varepsilon, \varepsilon) \to \mathbb{R}^{n\times n}$ as the product $\varphi(t) := A(t)B(t)$. Show that

$$\varphi'(t) = A'(t)B(t) + A(t)B'(t)$$

for every $t \in (-\varepsilon, \varepsilon)$.

Exercise 3.(Some Lie Algebras):

a) Let M, N be smooth manifolds and let $f : M \to N$ be a smooth map of constant rank r. By the constant rank theorem we know that the level set $L = f^{-1}(q)$ is a regular submanifold of M of dimension $\dim M - r$ for every $q \in N$. Show that one may canonically identify

$$T_pL \cong \ker d_pf$$

for every $p \in L = f^{-1}(q)$.

b) Use part a) to compute the Lie algebras of the Lie groups $O(n,\mathbb{R})$, $O(p,q)$, $U(n)$, $Sp(2n,\mathbb{C})$, $B(n)$ and $N(n)$ where $B(n)$ is the group of real invertible upper triangular matrices and $N(n)$ is the subgroup of $B(n)$ with only ones on the diagonal.

Exercise 4.(Easy Direction of Frobenius' Theorem):

Let M be a smooth manifold and let $\mathcal{D}$ be a distribution on M. Show that $\mathcal{D}$ is involutive if it is completely integrable.

Exercise 5.(Distributions and Lie Subalgebras):

a) Let M be a smooth manifold, $X, Y \in \mathrm{Vect}(M)$ vector fields on M, and $f, g \in C^\infty(M)$ smooth functions. Show that

$$[fX, gY] = fg[X, Y] + f(Xg)Y - g(Yf)X.$$

b) Show that the Lie algebra $\mathfrak{h}$ of a Lie subgroup H of a Lie group G determines a left-invariant involutive distribution.

Remark: Part a) is not necessarily needed for part b).

Exercise 6.(Functions with values in immersed submanifolds):

Let M', M, N be smooth manifolds and let $\iota\colon N \hookrightarrow M$ be an injective immersion, i.e. ι is an injective smooth map whose differential is injective. Further, let $f : M' \to M$ be a smooth map with $f(M) \subseteq \iota(N)$.

Show that $\iota^{-1} \circ f : M' \to N$ is smooth if it is continuous.

Due date: Thursday, 12/11/2020

Please, upload your solution via the SAM upload tool.

In order to access the website you will need a NETHZ-account and you will have to be connected to the ETH-network. From outside the ETH network you can connect to the ETH network via VPN. Here are instructions on how to do that.

Make sure that your solution is **one PDF file** and that its **file name** is formatted in the following way:

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solution_<number exercise sheet>_<last name>_<first name>.pdf
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For example: If your first name is Alice, your last name is Miller, and you want to hand-in your solution to Exercise Sheet 2, then you will have to upload your solution as **one** PDF file with the following file name:

```
solution_2_Miller_Alice.pdf
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Solutions that fail to comply with the above requirements will be ignored.