Exercise Sheet 5

Exercise 1.(Discrete Subgroups of $\mathbb{R}^n$):

Let $D < \mathbb{R}^n$ be a discrete subgroup. Show that there are $x_1, \ldots, x_k \in D$ such that

- $x_1, \ldots, x_k$ are linearly independent over $\mathbb{R}$, and
- $D = \mathbb{Z}x_1 \oplus \cdots \oplus \mathbb{Z}x_k$, i.e. $x_1, \ldots, x_k$ generate D as a $\mathbb{Z}$-submodule of $\mathbb{R}^n$.

Exercise 2.(Covering maps of Lie Groups):

Let G be a Lie group, let H be a simply connected topological space and let $p : H \to G$ be a covering map.

a) Show that there is a unique Lie group structure on H such that p is a smooth group homomorphism and that the kernel of p is a discrete subgroup of G.

b) Show that p is a local isomorphism of Lie groups and that dp is an isomorphism of Lie algebras when H is equipped with the Lie group structure from part a).

c) Let H, G be arbitrary Lie groups and let G be connected. Further, let $\varphi : H \to G$ be a Lie group homomorphism. Show that φ is a covering map if and only if $d\varphi : \mathfrak{h} \to \mathfrak{g}$ is an isomorphism.

Remark: Part a) and b) also work if H is not simply connected.

Exercise 3.(Abstract Subgroups as Lie Subgroups):

Let H be an abstract subgroup of a Lie group G and let $\mathfrak{h}$ be a subspace of the Lie algebra $\mathfrak{g}$ of G. Further let $U \subseteq \mathfrak{g}$ be an open neighborhood of $0 \in \mathfrak{g}$ and let $V \subseteq G$ be an open neighborhood of $e \in G$ such that the exponential map $\exp : U \to V$ is a diffeomorphism satisfying $\exp(U \cap \mathfrak{h}) = V \cap H$. Show that the following statements hold:

a) H is a Lie subgroup of G with the induced relative topology;

b) $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$;

c) $\mathfrak{h}$ is the Lie algebra of H.

Exercise 4.(Lie Group homomorphisms and their differentials):

Let G be a connected Lie group, let H be a Lie group and let $\varphi, \psi \colon G \to H$ be Lie group homomorphisms.

Show that $\varphi = \psi$ if and only if $d\varphi = d\psi$.

Exercise 5.(Surjectivity of the Matrix Exponential):

Let $\mathrm{Exp} : \mathfrak{gl}(n,\mathbb{R}) \cong \mathbb{R}^{n\times n} \to \mathrm{GL}(n,\mathbb{R})$ be the matrix exponential map given by the power series

$$\mathrm{Exp}(X) := \sum_{n=0}^{\infty} \frac{X^n}{n!}.$$

Consider the Lie subgroup of upper triangular matrices $N(n) < \mathrm{GL}(n,\mathbb{R})$ with its Lie algebra $\mathfrak{n}(n) < \mathfrak{gl}(n,\mathbb{R})$ of strictly upper triangular matrices; cf. exercise sheet 4 problem 3.

Show that $\mathrm{Exp}|_{\mathfrak{n}(n)} : \mathfrak{n}(n) \to N(n)$ is surjective.

Hint: Consider the partially defined matrix logarithm $\mathrm{Log} : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ given by

$$\mathrm{Log}(I + A) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{A^n}{n}.$$

Try to give answers to the following questions and then conclude:

What is its radius of convergence r about I? Why is it a right-inverse of Exp on the ball $B_r(I)$ of radius r about I? Why is there no problem for matrices that are in $N(n)$ but not in $B_r(I)$?

In order to answer the last question prove that $A^n = 0$ for all $A \in \mathfrak{n}(n)$.

Exercise 6.(Multiplication and exp):

Let G be a Lie group with Lie algebra $\mathfrak{g}$. Show that for all $X, Y \in \mathfrak{g}$ and small enough $t \in \mathbb{R}$

$$\exp(tX)\exp(tY) = \exp(t(X+Y) + O(t^2))$$

where $O(t^2)$ is a differentiable $\mathfrak{g}$-valued function such that $\frac{O(t^2)}{t^2}$ is bounded as $t \to 0$.

Due date: Thursday, 26/11/2020

Please, upload your solution via the SAM upload tool.

In order to access the website you will need a NETHZ-account and you will have to be connected to the ETH-network. From outside the ETH network you can connect to the ETH network via VPN. Here are instructions on how to do that.

Make sure that your solution is **one PDF file** and that its **file name** is formatted in the following way:

solution_<number exercise sheet>_<last name>_<first name>.pdf

For example: If your first name is Alice, your last name is Miller, and you want to hand-in your solution to Exercise Sheet 2, then you will have to upload your solution as **one** PDF file with the following file name:

solution_2_Miller_Alice.pdf

Solutions that fail to comply with the above requirements will be ignored.