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Introduction to Lie Groups — Autumn 2020

Exercise Sheet 7

Exercise 1.(Ideals and Quotients of Nilpotent Lie Algebras):

Let $\mathfrak{g}$ be a Lie algebra and let $\mathfrak{h} \trianglelefteq \mathfrak{g}$ be an ideal. We have already seen that, $\mathfrak{g}$ is solvable if and only if $\mathfrak{h}$ and $\mathfrak{g}/\mathfrak{h}$ are solvable.

Show that such a statement cannot hold for nilpotent $\mathfrak{g}$!

Exercise 2.(Adjoint of nilpotent elements):

Let $\mathfrak{g} \leq \mathfrak{gl}_n(\mathbb{C})$ be a Lie subalgebra.

Show that, if $X \in \mathfrak{g}$ is nilpotent then $\mathrm{ad}(X) \in \mathfrak{gl}(\mathfrak{g})$ is nilpotent.

Exercise 3.(Cartan's criterion for solvability):

Let $\mathfrak{g}$ be a Lie algebra with Killing form $B_{\mathfrak{g}}$.

Show that $\mathfrak{g}$ is solvable if and only if $B_{\mathfrak{g}}|_{\mathfrak{g}^{(1)} \times \mathfrak{g}^{(1)}} = 0$.

Hint: One direction is an easy verification. For the other direction you can use a theorem from class in conjunction with the fact that a Lie algebra $\mathfrak{g}$ with an ideal $\mathfrak{h} \trianglelefteq \mathfrak{g}$ is solvable if and only if both $\mathfrak{h}$ and $\mathfrak{g}/\mathfrak{h}$ are solvable.

Exercise 4.(Direct sums of simple ideals):

Let $\mathfrak{g} = \bigoplus_{i \in I} \mathfrak{g}_i$ be the direct sum of simple ideals. Then any ideal $\mathfrak{h} \trianglelefteq \mathfrak{g}$ is of the form $\mathfrak{h} = \bigoplus_{j \in J} \mathfrak{g}_j$ with $J \subset I$.

Remark: This implies immediately:

(i) Any semisimple Lie algebras has a finite number of ideals.

(ii) Any connected semisimple Lie group with finite center has a finite number of connected normal subgroups.

Exercise 5.(Characterization of Semi-Simplicity):

Let $\mathfrak{g}$ be a Lie algebra. Show that the following statements are equivalent:

(i) $\mathfrak{g}$ is semisimple;

(ii) $\mathfrak{g}$ has no non-trivial abelian ideals;

(iii) $\mathfrak{g}$ has no non-trivial solvable ideals.

Exercise 6.(Semi-Simple Lie Algebras equal their commutator):

Let $\mathfrak{g}$ be a Lie algebra. Show that if $\mathfrak{g}$ is semisimple then $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$.

This is the last exercise sheet and it will not be graded.

We hope that you have had a good time in our course and wish you the best of luck in the exam.

Merry christmas and a happy new year!

Alessandra and Yannick