Solution Exercise Sheet 1

Exercise 1.(Unitary Operators):

Let $\mathcal{H}$ be a Hilbert space and $U(\mathcal{H})$ its group of unitary operators. Show that the weak operator topology coincides with the strong operator topology on $U(\mathcal{H})$.

Solution: Recall that a sequence $(T_n)_{n\in\mathbb{N}} \subset U(\mathcal{H})$ of unitary operators converges to a unitary operator T with respect to the *weak operator topology* if

$$\lambda(T_n x) \to \lambda(Tx) \quad (n \to \infty)$$

for every linear functional $\lambda \in \mathcal{H}^*$ and every $x \in \mathcal{H}$.

A sequence $(T_n)_{n\in\mathbb{N}} \subset U(\mathcal{H})$ of unitary operators converges to a unitary operator T with respect to the *strong operator topology* if

$$T_n x \to Tx \quad (n \to \infty)$$

for every $x \in \mathcal{H}$.

In order to show that the weak operator topology coincides with the strong operator topology it will be sufficient to show that a sequence $(T_n)_{n\in\mathbb{N}} \subset U(\mathcal{H})$ converges with respect to the weak operator topology to $T \in U(\mathcal{H})$ if and only if $(T_n)_{n\in\mathbb{N}}$ converges with respect to the strong operator topology to T.

" $\Longleftarrow$ ": Let $T_n \to T$ strongly and let $\lambda \in \mathcal{H}^*, x \in \mathcal{H}$. Then because λ is continuous and $T_n x \to T_x$ we get

$$\lambda(T_n x) \to \lambda(Tx)$$

as $n \to \infty$.

" $\Longrightarrow$ ": Let $T_n \to T$ weakly and let $x \in \mathcal{H}$. We need to see that

$$\|T_n x - Tx\|^2 \to 0 \quad (n \to \infty).$$

We compute

$$\begin{aligned}
\|T_n x - Tx\|^2 &= \langle T_n x - Tx, T_n - Tx\rangle \\
&= \langle T_n x, T_n x\rangle - \langle T_n x, Tx\rangle - \langle Tx, T_n x\rangle + \langle Tx, Tx\rangle \\
&= \langle x, x\rangle - \langle T_n x, Tx\rangle - \langle Tx, T_n x\rangle + \langle x, x\rangle \\
&= 2\|x\|^2 - \left(\langle T_n x, Tx\rangle + \overline{\langle T_n x, Tx\rangle}\right) \\
&= 2\|x\|^2 - 2\mathfrak{Re}\left(\langle T_n x, Tx\rangle\right) \\
&\to 2\|x\|^2 - 2\|Tx\|^2 = 2\|x\|^2 - 2\|x\|^2 = 0 \quad (n \to \infty),
\end{aligned}$$

where we have used that T_n and T are unitary and that $\langle \cdot, Tx\rangle$ is a continuous linear functional.

□

Exercise 2.(Compact-Open Topology):

Let X, Y, Z be topological space, and denote by $C(Y,X) := \{f: Y \to X \text{ continuous}\}$ the set of continuous maps from Y to X. The set $C(Y,X)$ can be endowed with the *compact-open topology*, that is generated by the subbasic sets

$$S(K,U) := \{f \in C(Y,X) | f(K) \subseteq U\},$$

where $K \subseteq Y$ is compact and $U \subseteq X$ is open.

Prove the following useful facts about the compact-open topology.
If Y is locally compact, then:

a) The evaluation map $e\colon C(Y,X) \times Y \to X, e(f,y) := f(y)$, is continuous.

b) A map $f: Y \times Z \to X$ is continuous if and only if the map

$$\hat{f}: Z \to C(Y,X), \hat{f}(z)(y) = f(y,z),$$

is continuous.

Solution:

a) For $(f,y) \in C(Y,X) \times Y$ let $U \subset X$ be an open neighborhood of $f(y)$. Since Y is locally compact, continuity of f implies there is a compact neighborhood $K \subset Y$ of y such that $f(K) \subset U$. Then $S(K,U) \times K$ is a neighborhood of (f,y) in $C(Y,X) \times Y$ taken to U by e, so e is continuous at (f,y).

b) Suppose $f\colon Y \times Z \to X$ is continuous. To show continuity of $\hat{f}$ it suffices to show that for a subbasic set $S(K,U) \subset C(Y,X)$, the set $\hat{f}^{-1}(S(K,U)) = \{z \in Z | f(K,z) \subset U\}$ is open in Z. Let $z \in \hat{f}^{-1}(S(K,U))$. Since $f^{-1}(U)$ is an open neighborhood of the compact set $K \times \{z\}$, there exist open sets $V \subset Y$ and

$W \subset Z$ whose product $V \times W$ satisfies $K \times \{z\} \subset V \times W \subset f^{-1}(U)$. So W is a neighborhood of z in $\hat{f}^{-1}(S(K,U))$. (The hypothesis that Y is locally compact is not needed here.)

For the converse of b) note that f is the composition $Y \times Z \to Y \times C(Y,X) \to X$ of $\mathrm{Id} \times \hat{f}$ and the evaluation map, so part a) gives the result.

□

Exercise 3.(General Linear Group $\mathrm{GL}(n,\mathbb{R})$**):**

The general linear group

$$\mathrm{GL}(n,\mathbb{R}) := \{A \in \mathbb{R}^{n\times n} \,|\, \det A \neq 0\} \subseteq \mathbb{R}^{n\times n}$$

is naturally endowed with the subspace topology of $\mathbb{R}^{n\times n} \cong \mathbb{R}^{n^2}$. However, it can also be seen as a subset of the space of homeomorphisms of $\mathbb{R}^n$ via the injection

$$\begin{aligned} j\colon\ \mathrm{GL}(n,\mathbb{R}) &\to \mathrm{Homeo}(\mathbb{R}^n), \\ A &\mapsto (x \mapsto Ax). \end{aligned}$$

a) Show that $j(\mathrm{GL}(n,\mathbb{R})) \subset \mathrm{Homeo}(\mathbb{R}^n)$ is a closed subset, where $\mathrm{Homeo}(\mathbb{R}^n) \subset C(\mathbb{R}^n,\mathbb{R}^n)$ is endowed with the compact-open topology.

   Solution: Note that

   $$j(\mathrm{GL}(n,\mathbb{R})) = \{f \in \mathrm{Homeo}(\mathbb{R}^n) : f(\lambda x + y) = \lambda f(x) + f(y) \text{ for all } \lambda \in \mathbb{R}, x, y \in \mathbb{R}^n\}.$$

   Since evaluation is continuous also the maps

   $$\begin{aligned} F_{\lambda,x,y} : \mathrm{Homeo}(\mathbb{R}^n) &\to \mathbb{R}^n \\ f &\mapsto f(\lambda x + y) - \lambda f(x) + f(y) \end{aligned}$$

   are continuous for all $\lambda \in \mathbb{R}, x, y \in \mathbb{R}^n$.

   Thus,

   $$j(\mathrm{GL}(n,\mathbb{R})) = \bigcap_{\lambda \in \mathbb{R}, x,y \in X} F^{-1}_{\lambda,x,y}(0) \subset \mathrm{Homeo}(\mathbb{R}^n)$$

   is closed as the intersection of closed sets.

   □

b) If we identify $\mathrm{GL}(n,\mathbb{R})$ with its image $j(\mathrm{GL}(n,\mathbb{R})) \subset \mathrm{Homeo}(\mathbb{R}^n)$ we can endow it with the induced subspace topology. Show that this topology coincides with the usual topology coming from the inclusion $\mathrm{GL}(n,\mathbb{R}) \subset \mathbb{R}^{n\times n}$.

Solution: Consider the inclusions

$$\begin{aligned} i : \mathrm{GL}(n,\mathbb{R}) &\to \mathbb{R}^{n\times n}, \\ A &\mapsto \begin{pmatrix} | & & | \\ A\mathbf{e}_1 & \cdots & A\mathbf{e}_n \\ | & & | \end{pmatrix}, \end{aligned}$$

where $\mathbf{e}_1,\dots,\mathbf{e}_n$ denotes the standard basis of $\mathbb{R}^{n\times n}$.

Further, consider the maps

$$\begin{aligned} \varphi : \mathbb{R}^{n\times n} &\to C(\mathbb{R}^n,\mathbb{R}^n), \\ \begin{pmatrix} | & & | \\ \mathbf{v}_1 & \cdots & \mathbf{v}_n \\ | & & | \end{pmatrix} &\mapsto (\mathbf{x} \mapsto x_1\cdot\mathbf{v}_1 + \cdots + x_n\cdot\mathbf{v}_n), \end{aligned}$$

and

$$\begin{aligned} \psi : C(\mathbb{R}^n,\mathbb{R}^n) &\to \mathbb{R}^{n\times n}, \\ f &\mapsto \begin{pmatrix} | & & | \\ f(\mathbf{e}_1) & \cdots & f(\mathbf{e}_n) \\ | & & | \end{pmatrix}. \end{aligned}$$

It is easy to verify that these form the following commutative diagram.

$$\begin{tikzcd} & \mathrm{GL}(n,\mathbb{R}) \arrow[dl, hook, "i"'] \arrow[dr, hook, "j"] & \\ \mathbb{R}^{n\times n} \arrow[rr, bend left, "\varphi"] & & C(\mathbb{R}^n,\mathbb{R}^n) \arrow[ll, bend left, "\psi"] \end{tikzcd}$$

Since both topologies under consideration on $\mathrm{GL}(n,\mathbb{R})$ come from pulling back the topologies of $\mathbb{R}^{n\times n}$ resp. $C(\mathbb{R}^n,\mathbb{R}^n)$ via i resp. j they will coincide if we can show that the maps φ and ψ are continuous.

The map ψ is continuous because it is the product of the evaluation maps

$$\mathrm{ev}_{\mathbf{e}_i} : C(\mathbb{R}^n,\mathbb{R}^n) \to \mathbb{R}^n, \mathrm{ev}_{\mathbf{e}_i}(f) = f(\mathbf{e}_i)$$

$(i = 1,\dots,n)$.

Further, observe that the map

$$\mathrm{ev} \circ (\varphi \times \mathrm{Id}) : \mathbb{R}^{n\times n} \times \mathbb{R}^n \to \mathbb{R}^n, (A,x) \mapsto Ax$$

is continuous. This implies that φ is continuous.

$\square$

Hint: Exercise 2 can be useful here.

Exercise 4.(Isometry Group $\mathrm{Iso}(X)$**):**

Let (X, d) be a *compact* metric space. Recall that the isometry group of X is defined as

$$\mathrm{Iso}(X) = \{f \in \mathrm{Homeo}(X) : d(f(x), f(y)) = d(x, y) \quad \text{for all } x, y \in X\}.$$

Show that $\mathrm{Iso}(X) \subset \mathrm{Homeo}(X)$ is compact with respect to the compact-open topology.

Hint: Use the fact that the compact-open topology is induced by the metric of uniform-convergence and apply Arzelà–Ascoli's theorem.

Solution: The compact-open topology on $\mathrm{Homeo}(X)$ coincides with the topology induced by the metric of uniform convergence

$$d_\infty(f, g) = \sup\{d(f(x), g(x)) : x \in X\}.$$

Note that by Arzelà–Ascoli a family $\mathcal{F} \subseteq C(X, X)$ of continuous maps is compact if and only if $\mathcal{F}$ is equicontinuous, $\mathcal{F}_x = \{f(x) : f \in \mathcal{F}\}$ is relatively compact for every $x \in X$ and $\mathcal{F}$ is closed.

Equicontinuity of $\mathcal{F} := \mathrm{Iso}(X)$ is clear, because we are dealing with isometries. Moreover, $\mathcal{F}_x = \{f(x) : f \in \mathrm{Iso}(X)\} \subseteq X$ is a subset of a compact space, whence relatively compact. All that is left to check is that $\mathrm{Iso}(X)$ is closed.

Let $f \in C(X, X)$ and let $(f_n)_{n\in\mathbb{N}} \subset \mathrm{Iso}(X)$ be a sequence converging to it. Let $x, y \in X$ then

$$\begin{aligned}
0 &\leq \left|d(f(x), f(y)) - d(x, y)\right| \\
&= \left|d(f(x), f(y)) - d(f_n(x), f_n(y))\right| \\
&\leq \left|d(f(x), f(y)) - d(f_n(x), f(y))\right| + \left|d(f_n(x), f(y)) - d(f_n(x), f_n(y))\right| \\
&\leq d(f(x), f_n(x)) + d(f(y), f_n(y)) \to 0 \quad (n \to \infty).
\end{aligned}$$

Hence, f is an isometry as wished for. Because f was arbitrary this shows that $\mathrm{Iso}(X) \subseteq C(X, X)$ is closed. □

Exercise 5.(p**-adic Integers** $\mathbb{Z}_p$**):**

Let $p \in \mathbb{N}$ be a prime number. Recall that the p-adic integers $\mathbb{Z}_p$ can be seen as the subspace

$$\left\{(a_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} \mathbb{Z}/p^n\mathbb{Z} : a_{n+1} \equiv a_n \,(\mathrm{mod}\ p^n)\right\}$$

of the infinite product $\prod_{n\in\mathbb{N}} \mathbb{Z}/p^n\mathbb{Z}_p$ carrying the induced topology. Note that each

$\mathbb{Z}/p^n\mathbb{Z}$ carries the discrete topology and $\prod_{n\in\mathbb{N}} \mathbb{Z}/p^n\mathbb{Z}$ is endowed with the resulting product topology.

a) Show that the image of $\mathbb{Z}$ via the embedding

$$\begin{aligned} \iota : \mathbb{Z} &\to \mathbb{Z}_p, \\ x &\mapsto (x \,(\text{mod } p^n))_{n\in\mathbb{N}} \end{aligned}$$

is dense. In particular, $\mathbb{Z}_p$ is a compactification of $\mathbb{Z}$.

Solution: Let $(x_n) \in \mathbb{Z}_p$. A neighborhood basis of (x_n) is given by the sets

$$B_m((x_n)) = \{(y_n) \in \mathbb{Z}_p : x_1 = y_1, \ldots, x_m = y_m\}, \quad m \in \mathbb{N}.$$

Let $m \in \mathbb{N}$. We want to construct an integer $x \in \mathbb{Z}$ such that $\iota(x) \in B_m((x_n))$. It suffices to take a preimage $x \in \mathbb{Z}$ of $x_m \in \mathbb{Z}/p^m\mathbb{Z}$ under $\pi_m : \mathbb{Z} \to \mathbb{Z}/p^m\mathbb{Z}$. Then we clearly obtain

$$\begin{aligned} x_m &\equiv x (\text{mod } p^m), \\ x_{m-1} &\equiv x_m (\text{mod } p^{m-1}) \equiv x (\text{mod } p^{m-1}), \\ &\vdots \\ x_1 &\equiv x (\text{mod } p). \end{aligned}$$

That is $\iota(x) \in B_m((x_n))$.

$\square$

b) Show that the 2-adic integers $\mathbb{Z}_2$ are homeomorphic to the "middle thirds" cantor set C as defined in Exercise 6b).

Solution: We will prove that the map

$$\begin{aligned} \varphi : C &\to \mathbb{Z}_2, \\ \sum_{n=1}^{\infty} \varepsilon_n 3^{-n} &\mapsto \left(\sum_{k=1}^{n} \frac{\varepsilon_k}{2} \cdot 2^{k-1} \right)_{n\in\mathbb{N}} \end{aligned}$$

is a homeomorphism.

φ is well-defined because

$$\varphi\left(\sum_{n=1}^{\infty} \varepsilon_n 3^{-n} \right)_n \equiv \sum_{k=1}^{n} \frac{\varepsilon_k}{2} \cdot 2^{k-1} + \frac{\varepsilon_{n+1}}{2} \cdot 2^n \equiv \varphi\left(\sum_{n=1}^{\infty} \varepsilon_n 3^{-n} \right)_{n+1} \quad (\text{mod } 2^n).$$

By uniqueness of 2-adic expansions φ is injective.

φ is surjective because for every $(x_n)_{n\in\mathbb{N}} \in \mathbb{Z}_2$ we can find 2-adic expansions

$$x_n = a_0^{(n)} + a_1^{(n)} \cdot 2 + \cdots + a_{n-1}^{(n)} \cdot 2^{n-1}, \quad n \in \mathbb{N},$$

with unique $a_i^{(n)} \in \{0,1\}$. By the compatibility condition in $\mathbb{Z}_2$

$$x_n \equiv x_{n+1} (\operatorname{mod} 2^n)$$

we get that $a_i^{(n)} = a_i^{(n+1)}$ for every $i < n$. Hence, we can write

$$x_n = a_0 + a_1 \cdot 2 + \cdots + a_{n-1} \cdot 2^{n-1}, \quad n \in \mathbb{N},$$

with unique $a_i \in \{0,1\}$. Thus,

$$\varphi\left(\sum_{n=1}^{\infty} 2a_n 3^{-n}\right) = (x_n)_{n\in\mathbb{N}},$$

i.e. φ is surjective.

In order to prove that φ is continuous and open we first need to deduce the following neat relation: For every $c = \sum_{n=1}^{\infty} \varepsilon_n 3^{-n}, d = \sum_{n=1}^{\infty} \delta_n 3^{-n} \in C$

$$-\log_3 |d-c| \le \min\{k \in \mathbb{N} : \varepsilon_k \neq \delta_k\} \le -\log_3 |d-c| + 1.$$

Indeed, let $m = \min\{k \in \mathbb{N} : \varepsilon_k \neq \delta_k\}$. Then

$$\begin{aligned} |d-c| &= \left|(\delta_m - \varepsilon_m) \cdot 3^{-m} + \sum_{n=m+1}^{\infty} (\delta_n - \varepsilon_n) \cdot 3^{-n}\right| \\ &\ge \left|\underbrace{|\delta_m - \varepsilon_m|}_{=2} \cdot 3^{-m} - \left|\sum_{n=m+1}^{\infty} (\delta_n - \varepsilon_n) \cdot 3^{-n}\right|\right| \\ &\ge \frac{2}{3^m} - \sum_{n=m+1}^{\infty} |\delta_n - \varepsilon_n| \cdot 3^{-n} \\ &\ge \frac{2}{3^m} - \sum_{n=m+1}^{\infty} 2 \cdot 3^{-n} = \frac{2}{3^m} - \frac{1}{3^m} = 3^{-m}. \end{aligned}$$

Applying the logarithm to base 3 on both sides yields the first inequality.

The second inequality follows from the following easier computation.

$$\begin{aligned} |d-c| &= \left|\sum_{n=m}^{\infty} (\delta_n - \varepsilon_n) \cdot 3^{-n}\right| \le \sum_{n=m}^{\infty} 2 \cdot 3^{-n} = \frac{1}{3^{m-1}} \\ &\implies \log_3 |d-c| \le -m + 1. \end{aligned}$$

Now, let $c = \sum_{n=1}^{\infty} \varepsilon_n 3^{-n} \in C$ and consider a neighborhood $B_m(\varphi(c))$. Then

$$\begin{aligned}
&d = \sum_{n=1}^{\infty} \delta_n 3^{-n} \in \varphi^{-1}(B_m(\varphi(c)))\\
\iff &\sum_{k=1}^{l} \frac{\varepsilon_k}{2} \cdot 2^{k-1} = \sum_{k=1}^{l} \frac{\delta_k}{2} \cdot 2^{k-1}, \quad \forall 1 \leq l \leq m\\
\iff &\varepsilon_k = \delta_k, \quad \forall k = 1, \dots, m\\
\iff &\min\{k \in \mathbb{N} : \varepsilon_k \neq \delta_k\} \geq m+1
\end{aligned}$$

By the previously deduced relation this readily implies

$$B_{m+1}(\varphi(c)) \subset \varphi(C \cap (-3^{-m} + c, c + 3^{-m})) \subset B_m(\varphi(c)).$$

It follows that φ is continuous and open. $\square$

Exercise 6†.(Homeomorphism Group Homeo(X)):

a) Let X be a *compact* Hausdorff space. Show that $(\text{Homeo}(X), \circ)$ is a topological group when endowed with the compact-open topology.

Solution: Denote by $m : \text{Homeo}(X) \times \text{Homeo}(X) \to \text{Homeo}(X)$ the composition $m(f, g) = f \circ g$ and by $i : \text{Homeo}(X) \to \text{Homeo}(X)$ the inversion $i(f) = f^{-1}$. We need to see that m and i are continuous.

i) m is continuous: We want to show that m is continuous at any tuple $(f, g) \in \text{Homeo}(X) \times \text{Homeo}(X)$. Thus let $S(K, U) \ni f \circ g$ be a subbasis neighborhood of $f \circ g$, i.e. $K \subset X$ is compact and $U \subset X$ is open such that $f(g(K)) \subset U$. Observe that $g(K)$ is compact and is contained in $f^{-1}(U)$ which is open. Because X is (locally) compact we may find an open set $V \subset X$ with compact closure $\overline{V}$ such that

$$g(K) \subset V \subset \overline{V} \subset f^{-1}(U).$$

It is now easy to verify that $W := S(\overline{V}, U) \times S(K, V)$ is an open neighborhood of (f, g) such that $m(W) \subset S(K, U)$. Indeed, (f, g) is by construction of V contained in W and for any $(h_1, h_2) \in W$ we get

$$h_2(K) \subset V \subset \overline{V} \subset h_1^{-1}(U).$$

Hence, m is continuous at every point of $\text{Homeo}(X) \times \text{Homeo}(X)$.

ii) i is continuous: Let $f \in \mathrm{Homeo}(X)$, $K \subset X$ compact and $U \subset X$ open. Then

$$\begin{aligned} i(f) \in S(K,U) &\iff f^{-1}(K) \subset U \iff K \subset f(U) \\ &\iff f(U^c) = f(U)^c \subset K^c \iff f \in S(U^c, K^c). \end{aligned}$$

Observe that U^c is compact as a closed subset of the compact space X and that K^c is open as the complement of a (compact) closed set.

This shows that $i^{-1}(S(K,U)) = S(U^c, K^c)$ for every element $S(K,U)$ of a subbasis for the compact-open topology on $\mathrm{Homeo}(X)$, whence i is continuous.

□

b) The objective of this exercise is to show that $(\mathrm{Homeo}(X), \circ)$ will not necessarily be a topological group if X is only locally compact.

Consider the "middle thirds" Cantor set

$$C = \left\{ \sum_{n=1}^{\infty} \varepsilon_n 3^{-n} : \varepsilon_n \in \{0,2\} \text{ for each } n \in \mathbb{N} \right\} \subset [0,1]$$

in the unit interval. We define the sets $U_n = C \cap [0, 3^{-n}]$ and $V_n = C \cap [1 - 3^{-n}, 1]$. Further we construct a sequence of homeomorphisms $h_n \in \mathrm{Homeo}(C)$ as follows:

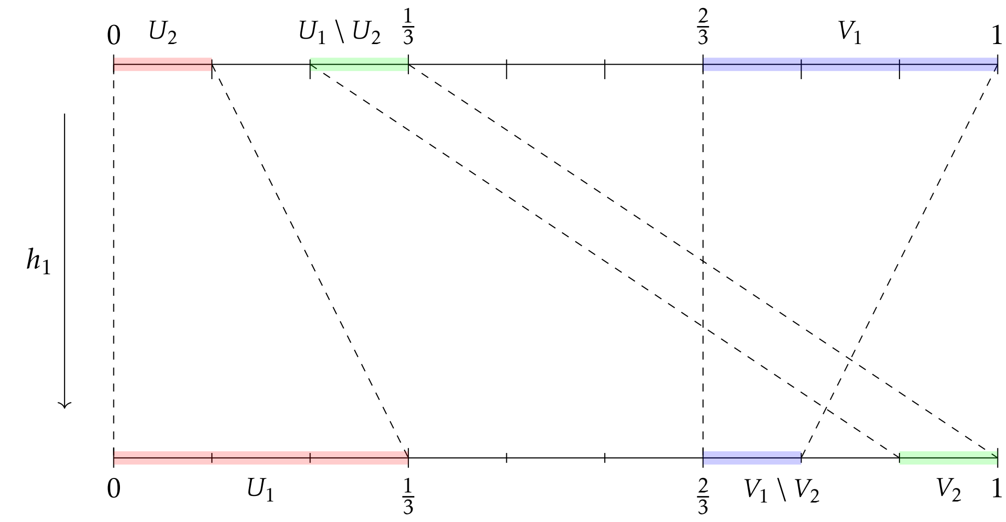
- $h_n(x) = x$ for all $x \in C \setminus (U_n \cup V_n)$,
- $h_n(0) = 0$,
- $h_n(U_{n+1}) = U_n$,
- $h_n(U_n \setminus U_{n+1}) = V_{n+1}$,
- $h_n(V_n) = V_n \setminus V_{n+1}$.

These restrict to homeomorphisms $h_n|_X$ on $X := C \setminus \{0\}$.

Show that the sequence $(h_n|_X)_{n \in \mathbb{N}} \subset \mathrm{Homeo}(X)$ converges to the identity on X but the sequence $((h_n|_X)^{-1})_{n \in \mathbb{N}} \subset \mathrm{Homeo}(X)$ of their inverses does not!

Remark: However, if X is locally compact and *locally connected* then $\mathrm{Homeo}(X)$ is a topological group.

Solution: The following picture gives a pictorial description of what h_1 does on the Cantor set C.



Since $h_n(0) = 0$ we obtain indeed a homeomorphism $h_n|_X \in \text{Homeo}(X)$ by restriction to $X = C \setminus \{0\}$. Let us first see that the sequence $(h_n|_X)_{n\in\mathbb{N}}$ indeed converges to $\text{Id} \in \text{Homeo}(X)$. For that let $S(K,U)$ be a subbasis neighborhood of Id, i.e. K is a compact subset of X contained in some open set $U \subset X$. Therefore we can find an $M \in \mathbb{N}$ such that U_M and K are disjoint.

If $1 \notin K$ then there is also an $N \geq M$ such that V_n and K are disjoint. In this case $h_n|K$ is the identity and hence in $S(K,U)$ for all $n \geq N$.

If $1 \in K$ then there is an $N \geq M$ such that V_N is contained in U. Consequently, we have

$$h_n(K \setminus V_n) = K \setminus V_n, \quad h_n(K \cap V_n) \subset V_n \subset V_N \subset U,$$

for all $n \geq N$.

In any case the sequence $(h_n|_X)_{n\in\mathbb{N}}$ will be in $S(K,U)$ for large enough n such that $\lim_{n\to\infty} h_n|_X = \text{Id}$. On the other hand $h_n^{-1}(1) \in U_n$ for every $n \in \mathbb{N}$ such that $\lim_{n\to\infty} h_n^{-1}(1) = 0$. Thus the sequence $(h_n^{-1}|_X)_{n\in\mathbb{N}}$ certainly does not converge to Id.

Remark: Note that we actually needed to remove 0 from C for this construction to work. In fact, the sequence h_n does not converge to Id in Homeo(C):

Let $K = [0,1/9] \cap C, U = [0,1/2) \cap C$. Then $S(K,U)$ is again a neighborhood of Id. However, $U_n \subset K$ for every $n \geq 2$ and $V_{n+1} \subset U^c$ which implies that

$$h_n(U_n \setminus U_{n+1}) \subset U^c,$$

i.e. $h_n \notin S(K,U)$.

□

c) Let $\mathbb{S}^1 \subset \mathbb{C}\setminus\{0\}$ denote the circle. Show that Homeo$(\mathbb{S}^1)$ is not locally compact.

Remark: In fact, Homeo(M) is not locally compact for any manifold M.

Solution: We will prove a more general fact, namely that Homeo(M) is not locally compact for any compact manifold M. Note that we can think of M as a compact metric space (M,d) by Urysohn's metrization theorem. In the case when M is a smooth manifolds this is even easier to see by endowing it with a Riemannian metric. This puts us now in the favorable position of being able to identify the compact-open topology on Homeo(X) with the topology of uniform convergence.

We denote by

$$d_\infty(f,g) := \sup\{d(f(x),g(x)) : x \in M\}$$

the metric of uniform convergence on Homeo(M). Further denote by $B_f^\infty(r)$ the ball of radius $r > 0$ about a homeomorphism $f \in \mathrm{Homeo}(M)$. In order to show that Homeo(M) is not locally compact we will construct in every $\varepsilon > 0$ ball about the identity $B_{\mathrm{Id}}^\infty(\varepsilon)$ a sequence of homeomorphisms $(f_k)_{k\in\mathbb{N}}$ with no convergent subsequence.

Let $\varepsilon > 0$ and denote $B = B_{\mathrm{Id}}^\infty(\varepsilon)$. Further, let $x_0 \in M$ and choose a coordinate chart $\varphi : U \subset B_{\varepsilon/2}(x_0) \to \mathbb{R}^n$ centered at x_0 (i.e. $\varphi(x_0) = 0$) contained in the $\varepsilon/2$-ball $B_{\varepsilon/2}(x_0)$ about x_0 in M. Consider the homeomorphisms

$$\psi_k : \overline{B_1}(0) \to \overline{B_1}(0), x \mapsto \|x\|^k x$$

on the closed unit ball $\overline{B_1}(0)$ in $\mathbb{R}^n$ which fix $0 \in \mathbb{R}^n$ and the boundary n-sphere pointwise. Note that the sequence $(\psi_k)_{k\in\mathbb{N}}$ converges pointwise to

$$\psi_\infty = \begin{cases} x, & \text{if } x \in \partial B_1(0), \\ 0, & \text{if } x \in B_1(0). \end{cases}$$

Now, define

$$f_k(x) := \begin{cases} x, & \text{if } x \notin \varphi^{-1}(B_1(0)), \\ \varphi^{-1}(\psi_k(\varphi(x))), & \text{if } x \in \varphi^{-1}(B_1(0)). \end{cases}$$

It is easy to see that the maps $f_k : M \to M$ are indeed homeomorphisms: $f_k|_{\varphi^{-1}(\overline{B_1}(0))^c} = \mathrm{Id} : \varphi^{-1}(\overline{B_1}(0))^c \to \varphi^{-1}(\overline{B_1}(0))^c$ is a homeomorphism, $\varphi^{-1} \circ \psi_k \circ \varphi : \varphi^{-1}(\overline{B_1}(0)) \to \varphi^{-1}(\overline{B_1}(0))$ is a homeomorphism and both coincide on $\varphi^{-1}(\partial B_1(0))$.

Further, the homeomorphisms f_k map the $\varepsilon/2$-ball $B_{\varepsilon/2}(x_0)$ to itself and fix x_0. Therefore,

$$d(f_k(x),x) \le d(f_k(x), \underbrace{f_k(x_0)}_{=x_0}) + d(x_0,x) < \varepsilon,$$

for every $x \in B_{\varepsilon/2}(x_0)$, and clearly $f_k(x) = x$ for every $x \notin B_{\varepsilon/2}(x_0)$. Hence, the sequence $(f_k)_{k\in\mathbb{N}}$ is in $B_\varepsilon^\infty(\mathrm{Id})$.

However, the sequence $(f_k)_{k\in\mathbb{N}}$ converges pointwise to

$$f_\infty(x) = \begin{cases} x, & \text{if } x \notin \varphi^{-1}(B_1(0)), \\ x_0, & \text{if } x \in \varphi^{-1}(B_1(0)), \end{cases}$$

If there were a subsequence $(f_{k_l})_{l\in\mathbb{N}}$ converging to some $f \in \mathrm{Homeo}(M)$ uniformly then this sequence would also converge pointwise to f, i.e. f needs to coincide with f_∞. But f_∞ is not even continuous which contradicts our assumption of $f \in \mathrm{Homeo}(M)$. Therefore $(f_k)_{k\in\mathbb{N}} \subset B_\varepsilon^\infty(\mathrm{Id})$ has no uniformly convergent subsequences.

$\square$