Solution Exercise Sheet 2

Exercise 1. (Identity Neighborhoods Generate Connected Groups):

Let $G$ be a connected topological group, $U \subset G$ an open neighborhood of the identity and $U^n := \{g_1 \cdots g_n | g_1, \ldots, g_n \in U\}$. Show that $G = \bigcup_{n=1}^{\infty} U^n$.

Hint: You may assume that $g^{-1} \in U$ for every $g \in U$. Why?

Solution: By replacing $U$ with $U \cap U^{-1}$ if necessary we may assume that $U$ is a symmetric neighborhood $U = U^{-1}$ of the identity $e \in G$.

Observe that $H = \bigcup_{n=1}^{\infty} U^n$ is a group. Indeed, for every $g_1 \cdots g_n, h_1 \cdots h_m \in H$ also

$$(g_1 \cdots g_n) \cdot (h_1 \cdots h_m)^{-1} = g_1 \cdots g_n \cdot h_m^{-1} \cdots h_1^{-1} \in U^{n+m} \subset H.$$

Further, $U$ is open and therefore also every

$$U^n = \bigcup_{g \in U^{n-1}} U \cdot g \subset G$$

is open as the union of open sets. Recall that right translation by group elements is a homeomorphism.

Hence, also $H = \bigcup_{n=1}^{\infty} U^n$ is open, i.e. $H \subset G$ is an open subgroup. In the lecture we have learned that open subgroups are always closed. Because $G$ is connected we have therefore $H = \emptyset$ or $H = G$. Since $H$ is non-empty the assertion follows.

Exercise 2. (Transitive Group Actions):

Let $G$ be a topological group, $X$ a topological space and $\mu : G \times X \to X$ a continuous transitive group action, i.e. for any two $x, y \in X$ there is $g \in G$ such that $\mu(g, x) = g \cdot x = y$.

a) Show that if $G$ is compact then $X$ is compact.

b) Show that if $G$ is connected then $X$ is connected.

Solution: Let $x_0 \in X$ and consider the map

$$\varphi : G \to X,$$

$$g \mapsto \mu(g, x_0).$$

Because $\mu$ is a continuous action the map $\varphi$ is continuous too. Further the action
\( \mu \) is transitive, i.e. for every \( y \in X \) there is a \( g \in G \) such that \( \mu(g, x_0) = y \). In other words, \( \varphi \) is surjective.

Part a) follows from the fact that \( X = \varphi(G) \) is compact as the image of a compact group.

Part b) follows from the fact that continuous maps send connected components to connected components and again that \( \varphi(G) = X \).

Exercise 3. (Examples of Haar Measures):

a) Let us consider the three-dimensional Heisenberg group \( H = \mathbb{R} \rtimes_{\eta} \mathbb{R}^2 \), where \( \eta : \mathbb{R} \to \text{Aut}(\mathbb{R}^2) \) is defined by

\[
\eta(x) \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} y \\ z + xy \end{pmatrix},
\]

for all \( x, y, z \in \mathbb{R} \). Thus the group operation is given by

\[(x_1, y_1, z_1) \ast (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2 + x_1 y_2)\]

and it is easy to see that it can be identified with the matrix group

\[
H \cong \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}
\]

Verify that the Lebesgue measure is the Haar measure of \( \mathbb{R} \rtimes_{\eta} \mathbb{R}^2 \) and that the group is unimodular.

Solution: Denote by \( \mu \) the measure on \( H \) induced by the Lebesgue measure on \( \mathbb{R}^3 \). In order to show that \( \mu \) is unimodular we need to see that

\[\mu(\lambda(h)f) = \mu(f) = \mu(\rho(h)f)\]

for every \( f \in C_c(H), h \in H \).

Let \( h_1 = (x_1, y_1, z_1) \in H \) and \( f \in C_c(H) \). We compute
\[
\int (\lambda(h_1^{-1}) f)(x_2, y_2, z_2) \, dx_2 \, dy_2 \, dz_2
= \int f(x_1 + x_2, y_1 + y_2, z_1 + z_2 + x_1 y_2) \, dx_2 \, dy_2 \, dz_2
\]
\text{Fubini = } \int f(x_1 + x_2, y_1 + y_2, z_2 + (z_1 + x_1 y_2)) \, dz_2 \, dx_2 \, dy_2
\text{transl. inv. = } \int f(x_1, y_1, y_2, z_2) \, dx_2 \, dy_2 \, dz_2
\text{F. & t.i. = } \int f(x_1, y_2, z_2) \, dx_2 \, dy_2 \, dz_2.
\]

This shows left-invariance.

\[
\int (p(h_1) f)(x_2, y_2, z_2) \, dx_2 \, dy_2 \, dz_2
= \int f(x_2 + x_1, y_2 + y_1, z_2 + z_1 + x_2 y_1) \, dx_2 \, dy_2 \, dz_2
\]
\text{Fubini = } \int f(x_1 + x_2, y_1 + y_2, z_2 + (z_1 + x_2 y_1)) \, dz_2 \, dx_2 \, dy_2
\text{transl. inv. = } \int f(x_1 + x_2, y_1 + y_2, z_2) \, dz_2 \, dx_2 \, dy_2
\text{F. & t.i. = } \int f(x_1, y_1, y_2, z_2) \, dx_2 \, dy_2 \, dz_2
\text{F. & t.i. = } \int f(x_1, y_2, z_2) \, dx_2 \, dy_2 \, dz_2.
\]

This shows right-invariance. Therefore \( \mu \) is a left- and right-invariant Haar measure on \( H \) and \( H \) is unimodular.

\[ \square \]

b) Let
\[
P = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a, b \in \mathbb{R}, a \neq 0 \right\}.
\]

Show that \( \frac{da}{a^2} \, db \) is the left Haar measure and \( da \, db \) is the right Haar measure.

In particular, \( P \) is not unimodular.

\textbf{Solution:} Let \( \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \in P \) and \( f \in C_c(P) \). We compute
\[ \int \left( \lambda \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \right) \begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix} dx dy = \int f \begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix} dx dy = \int f \begin{pmatrix} ax & ay + bx^{-1} \\ 0 & a^{-1}x^{-1} \end{pmatrix} a^2 \frac{dx}{(ax)^2} dy = \ldots \]

we change coordinates to \( \tilde{x} = ax, \tilde{y} = ay \) which has Jacobi determinant \( a^2 \)

\[ \ldots = \int f \begin{pmatrix} \tilde{x} & \tilde{y} + ab\tilde{x}^{-1} \\ 0 & \tilde{x}^{-1} \end{pmatrix} d\tilde{x} d\tilde{y} = \int f \begin{pmatrix} \tilde{x} & \tilde{y} + ab\tilde{x}^{-1} \\ 0 & \tilde{x}^{-1} \end{pmatrix} d\tilde{y} d\tilde{x} = \int f \begin{pmatrix} \tilde{x} & \tilde{y} \\ 0 & \tilde{x}^{-1} \end{pmatrix} d\tilde{x} d\tilde{y}. \]

This shows left-invariance for the measure \( \frac{d\tilde{x}}{\tilde{x}^2} d\tilde{y} \) as claimed.

We will now see that \( dadb \) is right-invariant:

\[ \int \left( \rho \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \right) \begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix} dx dy = \int f \begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix} dx dy = \int f \begin{pmatrix} ax & bx + a^{-1}y \\ 0 & a^{-1}x^{-1} \end{pmatrix} dx dy = \ldots \]

we change coordinates to \( \tilde{x} = ax, \tilde{y} = a^{-1}y \) which has Jacobi determinant 1

\[ \ldots = \int f \begin{pmatrix} \tilde{x} & ba^{-1}\tilde{x} + \tilde{y} \\ 0 & \tilde{x}^{-1} \end{pmatrix} d\tilde{x} d\tilde{y} = F \& t.i \]

\[ = \int f \begin{pmatrix} \tilde{x} & \tilde{y} \\ 0 & \tilde{x}^{-1} \end{pmatrix} d\tilde{x} d\tilde{y} \]

This shows right-invariance. Since both measures clearly do not coincide \( P \) is not unimodular.

\[ \square \]

c) Let \( G := \text{GL}_n(\mathbb{R}) \subseteq \mathbb{R}^{n^2} \) denote the group of invertible matrices over \( \mathbb{R} \). Let \( \lambda_n \)
denote the Lebesgue measure on $\mathbb{R}^{n^2}$. Prove that

$$dm(x) := |\det x|^{-n} d\lambda_{n^2}(x)$$

defines a bi-invariant (i.e. left- and right-invariant) Haar measure on $G$.

**Solution:** As $GL_n(\mathbb{R}) = \det^{-1}(\mathbb{R} \setminus \{0\})$ is open in $\mathbb{R}^{n^2}$, $\lambda_{n^2}|_{GL_n(\mathbb{R})}$ assigns non-zero measure to non-empty open and finite measure to compact subsets of $GL_n(\mathbb{R})$ (if $K \subseteq GL_n(\mathbb{R})$ is compact in $\mathbb{R}^{n^2}$, then $U \cap GL_n(\mathbb{R}) := \{ U \cap GL_n(\mathbb{R}); U \in \mathcal{U} \}$ is an open cover of $K$ in $GL_n(\mathbb{R})$, thus it admits a finite subcover and hence so does $U$). As det is continuous and does not vanish on $GL_n(\mathbb{R})$, the above also holds for $dm(g) := |\det g|^{-n} d\lambda_{n^2}(g)$.

It remains to show that $m$ is invariant. To this end we note that for $g \in GL_n(\mathbb{R})$, if $g = (g_1, \ldots, g_n)$ and $h \in GL_n(\mathbb{R})$, then

$$hg = (hg_1, \ldots, hg_n) \quad (g \in \text{Mat}_n(\mathbb{R})),$$

so that the left-action of $h$ on $GL_n(\mathbb{R})$ can be viewed as a restriction of a diagonal matrix $\text{diag}(h, \ldots, h) \in \mathbb{R}^{n^2 \times n^2}$ acting on a subset of $\mathbb{R}^{n^2}$. Let $f \in C_c(GL_n(\mathbb{R}))$, then

$$\int_{\mathbb{R}^{n^2}} 1_{GL_n(\mathbb{R})}(g) f(hg) |\det g|^{-n} d\lambda_{n^2}(g)$$

$$= \int_{\mathbb{R}^{n^2}} 1_{hGL_n(\mathbb{R})}(hg) f(hg) |\det hg|^{-n} |\det h|^{-n} d\lambda_{n^2}(g)$$

$$= \int_{\mathbb{R}^{n^2}} 1_{GL_n(\mathbb{R})}(hg) f(hg) |\det hg|^{-n} d\lambda_{n^2}(g)$$

$$= \int_{\mathbb{R}^{n^2}} 1_{GL_n(\mathbb{R})}(g) f(g) |\det g|^{-n} d\lambda_{n^2}(g),$$

where in the end we used the substitution formula for the map $\text{diag}(h, \ldots, h)$. This proves that $m$ is a left Haar measure on $GL_n(\mathbb{R})$. The measure is also right-invariant, because the map

$$g \mapsto \begin{pmatrix} g_1 h \\ \vdots \\ g_n h \end{pmatrix}$$

does also have Jacobian $|\det h|^n$ (for example because $gh = (h'g')^t$ and the Jacobian of transposition – being an idempotent map – is equal to 1). Thus $GL_n(\mathbb{R})$ is unimodular.

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d) Let $G = SL_n(\mathbb{R})$ denote the group of matrices of determinant 1 in $\mathbb{R}^{n \times n}$. For a Borel subset $B \subseteq SL_n(\mathbb{R})$ define

$$m(B) := \lambda_{n^2}\{tg; g \in B, t \in [0, 1]\}.$$
Show that $m$ is a well-defined bi-invariant Haar measure on $\text{SL}_n(\mathbb{R})$.

**Solution:** Let us first check that for any Borel subset $B \subseteq \text{SL}_n(\mathbb{R})$ the cone

$$C(B) = \{tb : b \in B, t \in [0,1]\}$$

is a Borel subset of $\mathbb{R}^{n^2}$. To this end we note first that

$$C(B) = C'(B) \cup \{0\},$$

where

$$C'(B) = \{tb : b \in B, t \in (0,1]\}.$$

It clearly suffices to show that $C'(B)$ is Borel. To this end let

$$\text{GL}^1_n(\mathbb{R}) = \{g \in \text{GL}_n(\mathbb{R}); |\det g| = 1\}.$$

Note that $\text{GL}^1_n(\mathbb{R}) \cong \text{SL}_n(\mathbb{R}) \rtimes C_2$, where $C_2$ is the group with two elements. As $\text{GL}^1_n(\mathbb{R})$ is homeomorphic to a disjoint union of two copies of $\text{SL}_n(\mathbb{R})$, $B$ is Borel in $\text{GL}^1_n(\mathbb{R})$. Define

$$\Psi : \text{GL}_n(\mathbb{R}) \rightarrow \text{GL}^1_n(\mathbb{R}), \quad g \mapsto \frac{1}{\sqrt{|\det g|}} g.$$

This is a Borel map and therefore

$$C'(B) = \Psi^{-1}(B) \cap \text{det}^{-1}(0,1]$$

is measurable.

The final claim now follows immediately from the argument in part [c]) which realizes the action of an element $g \in \text{SL}_n(\mathbb{R})$ on $\mathbb{R}^{n^2}$ as a diagonal action of $n$ copies of $g$, together with the fact that $\Phi \lambda_n^2 = |\det \Phi| \lambda_n^2$ for linear $\Phi$, $\det g = 1$, $C(gB) = gC(B)$ and $C(Bg) = C(B)g$ for all $g \in \text{SL}_n(\mathbb{R})$ and $B \subseteq \text{SL}_n(\mathbb{R})$ Borel.

e) Let $G$ denote the $ax + b$ group defined as

$$G = \left\{ \begin{pmatrix} a & b \\ 1 \end{pmatrix} ; a \in \mathbb{R}^\times, b \in \mathbb{R} \right\}.$$

Note that every element in $G$ can be written in a unique fashion as a product of the form:

$$\begin{pmatrix} a & b \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

where $\alpha \in \mathbb{R}^\times$ and $\beta \in \mathbb{R}$, which yields a coordinate system $\mathbb{R}^\times \times \mathbb{R} \leftrightarrow G$. Prove
that 

\[ dm(\alpha, \beta) = \frac{1}{|\alpha|} \, d\alpha \, d\beta \]

defines a left Haar measure on \( G \). Calculate \( \Delta_G(\alpha, \beta) \) for \( \alpha \in \mathbb{R}^x \) and \( \beta \in \mathbb{R} \).

**Solution:** We use the coordinate system \( \varphi : \text{Aff}_1(\mathbb{R}) \ni (a, b) \mapsto (a, a^{-1} b) \in \mathbb{R}^x \times \mathbb{R} \). On \( \mathbb{R}^x \times \mathbb{R} \) we define the measure \( d\nu(\alpha, \beta) := \frac{1}{|\alpha|} \, d\alpha \, d\beta \) and we claim that \( \varphi^{-1}, \nu \) is a left-Haar measure on \( \text{Aff}_1(\mathbb{R}) \).

Let \( f \in C_c(\text{Aff}_1(\mathbb{R})) \) and denote \( \psi(\alpha) := x\alpha \), then for left-translation – indicated by subscript – follows

\[
(\varphi^{-1}, \nu(f(x, y)_{1/1})) = \int_{\mathbb{R}^x} \left( \int_{\mathbb{R}^x} \frac{f(x, y)_{1/1} \circ \varphi^{-1}(\alpha, \beta)}{|\alpha|} \, d\beta \right) \, d\alpha
\]

\[
= \int_{\mathbb{R}^x} \left( \int_{\mathbb{R}^x} \frac{f \circ \varphi^{-1}(x\alpha, \beta + (x\alpha)^{-1}y)}{|\alpha|} \, d\beta \right) \, d\alpha
\]

(\text{trans. inv.)}

\[
= \int_{\mathbb{R}^x} \left( \int_{\mathbb{R}^x} \frac{f \circ \varphi^{-1}(x\alpha, \beta)}{|\alpha|} \, d\beta \right) \, d\alpha
\]

(\psi'(x) = x)

\[
\psi(\mathbb{R}^x) = \mathbb{R}^x
\]

and thus we have indeed found a left Haar measure. For right translation – indicated by superscript – follows

\[
(\varphi^{-1}, \nu(f(x, y)_{1/1})) = \int_{\mathbb{R}^x} \left( \int_{\mathbb{R}^x} \frac{f(x, y)_{1/1} \circ \varphi^{-1}(\alpha, \beta)}{|\alpha|} \, d\beta \right) \, d\alpha
\]

\[
= \int_{\mathbb{R}^x} \left( \int_{\mathbb{R}^x} \frac{f \circ \varphi^{-1}(x\alpha, x^{-1} \beta + x^{-1}y)}{|\alpha|} \, d\beta \right) \, d\alpha
\]

(\text{trans. inv.)}

\[
= \int_{\mathbb{R}^x} \left( \int_{\mathbb{R}^x} \frac{f \circ \varphi^{-1}(x\alpha, x^{-1} \beta)}{|\alpha|} \, d\beta \right) \, d\alpha
\]

(\text{subst. } \beta \mapsto x\beta)

\[
= \int_{\mathbb{R}^x} |x| \left( \int_{\mathbb{R}^x} \frac{f \circ \varphi^{-1}(x\alpha, \beta)}{|\alpha|} \, d\beta \right) \, d\alpha
\]

(as above)

\[
= |x| (\varphi^{-1}, \nu(f)).
\]

Hence \( \Delta_{\text{Aff}_1(\mathbb{R})}(x, y)_{1/1} = |x|^{-1} \).

**Exercise 4. (Haar Measure and Transitive Actions):**

Let \( G \) be a locally compact Hausdorff group and let \( X \) be a topological space.
Suppose that $G$ acts on $X$ continuously and transitively. Let $o \in X$, and denote $\pi : G \to X, g \mapsto g \cdot o$. Further, let

$$H := \text{Stab}(o) = \{ h \in G | h \cdot o = o \}$$

be the stabilizer of $o$.

Suppose there is a continuous section $\sigma : X \to G$ of $\pi$, i.e. $\pi \circ \sigma = \text{Id}_X$.

a) Show that $\psi : X \times H \to G, (x,h) \mapsto \sigma(x)h$ is a homeomorphism.

**Hint:** Find a continuous inverse!

**Solution:** We define $\varphi : G \to X \times H$ via

$$\varphi(g) := (\pi(g), \sigma(\pi(g))^{-1}g)$$

for all $g \in G$.

Note that $\sigma(\pi(g)) \cdot o = \pi(\sigma(\pi(g))) = \pi(g) = g \cdot o$,

whence $\sigma(\pi(g))^{-1}g \cdot o = o$ and $\sigma(\pi(g))^{-1}g \in H = \text{Stab}(o)$. This shows that $\varphi$ is well-defined. Moreover, $\varphi$ is continuous as a composition of continuous functions.

We will now show that $\varphi$ is the inverse of $\psi$, i.e. $\psi \circ \varphi = \text{Id}_G$ and $\varphi \circ \psi = \text{Id}_{X \times H}$.

Let $g \in G$. We compute:

$$\psi(\varphi(g)) = \psi(\pi(g), \sigma(\pi(g))^{-1}g) = \sigma(\pi(g))\sigma(\pi(g))^{-1}g = g.$$ 

Let $x \in X, h \in H$. We compute:

$$\varphi(\psi(x,h)) = \varphi(\sigma(x)h) = (\pi(\sigma(x)h), \sigma(\pi(\sigma(x)h))^{-1}\sigma(x)h) = (\sigma(x)h \cdot o, \sigma(\sigma(x)h \cdot o)^{-1}\sigma(x)h) = (\sigma(x) \cdot o, \sigma(\sigma(x) \cdot o)^{-1}\sigma(x)h) = (x, \sigma(x)^{-1}\sigma(x)h) = (x, h).$$

b) Suppose there is a (left) Haar measure $\nu$ on $H$ and suppose there is a left $G$-invariant Borel regular measure $\lambda$ on $X$. 

Show that the push-forward measure $\psi_*(\lambda \otimes \nu)$ is a (left) Haar measure on $G$.

**Solution:** All we need to see is that the push-forward measure $\mu = \psi_*(\lambda \otimes \nu)$ is left $G$-invariant.

Let $f \in C_c(G)$ and $g_0 \in G$. We compute:

\[
\int_G f(g_0 g) \, d\mu(g) = \int_{X \times H} f(g_0 \psi(x, h)) \, d(\lambda \otimes \nu)(x, h)
\]

(Fubini) \[
= \int_X \int_H f(g_0 \sigma(x) h) \, d\nu(h) \, d\lambda(x)
\]

(left invariance of $\nu$) \[
= \int_X \int_H f(\sigma(g_0 \cdot x) \sigma(g_0 \cdot x)^{-1} g_0 \sigma(x) h) \, d\nu(h) \, d\lambda(x)
\]

\[
= \int_X \int_H f(\sigma(x) h) \, d\nu(h) \, d\lambda(x)
\]

(left G-invariance of $\lambda$) \[
= \int_G f(g) \, d\mu(g)
\]

\[\square\]

**c)** Find a Haar measure on $\text{Iso}(\mathbb{R}^2)$.

**Solution:** Note that $\text{Iso}(\mathbb{R}^2)$ acts continuously and transitively on $\mathbb{R}^2$. Indeed, any translation $T_x: \mathbb{R}^2 \to \mathbb{R}^2, y \mapsto x + y$ ($x \in \mathbb{R}^2$) is a Euclidean isometry, that maps 0 to $x$.

In fact, this construction yields a continuous section $\sigma: \mathbb{R}^2 \to \text{Iso}(\mathbb{R}^2), x \mapsto T_x$, and we can apply part [b]. Indeed, the Lebesgue measure $\lambda$ on $\mathbb{R}^2$ is $\text{Iso}(\mathbb{R}^2)$-invariant, and it can be shown that the stabilizer of 0 is given by the orthogonal group $O(2, \mathbb{R})$. By part [b], a Haar measure on $\text{Iso}(\mathbb{R}^2)$ is given by the push-forward measure $\mu := \psi_*(\lambda \otimes \nu)$ where $\nu$ is a left Haar measure on $O(2, \mathbb{R})$.

We will apply part [b] again to compute a Haar measure on $O(2, \mathbb{R})$ more explicitly. Observe that $O(2, \mathbb{R})$ acts transitively on the group with two elements $\{\pm 1\}$ via $k \ast \varepsilon := \det(k) \cdot \varepsilon$ for every $k \in O(2, \mathbb{R}), \varepsilon \in \{\pm 1\}$. We obtain a surjective map $p = \det: O(2, \mathbb{R}) \to \{\pm 1\}, k \mapsto \det(k) \cdot 1 = \det(k)$. A section $\tau: \{\pm 1\} \to O(2, \mathbb{R})$ of det is given by

\[
\tau(\varepsilon) = \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix},
\]

which is continuous because $\{\pm 1\}$ carries the discrete topology. The stabilizer of 1 is then $\det^{-1}(1) \cap O(2, \mathbb{R}) = \text{SO}(2, \mathbb{R})$ and the usual Lebesgue measure on $[0, 2\pi)$ pushes-forward to a left Haar measure $\xi = \varphi_*(\lambda|_{[0, 2\pi)})$ on $\text{SO}(2, \mathbb{R})$ via...
the map

$$
\varphi : [0, 2\pi) \rightarrow \text{SO}(2, \mathbb{R}),
$$

$$
\theta \mapsto \begin{pmatrix}
\cos(\theta) & -\sin(\theta) \\
\sin(\theta) & \cos(\theta)
\end{pmatrix}
$$

Clearly, an invariant measure on \{\pm 1\} is given by the counting measure. Therefore, a left Haar measure \( \nu \) on \( O(2, \mathbb{R}) \) is given by

$$
\int_{O(2, \mathbb{R})} f(k) d\nu(k) = \sum_{\varepsilon = \pm 1} \int_{0}^{2\pi} f\left( \tau(\varepsilon) \cdot \varphi(\theta) \right) d\theta
$$

for every \( f \in C_c(O(2, \mathbb{R})) \).

Putting everything together we obtain

$$
\int_{\text{Iso}(\mathbb{R}^2)} f(g) d\mu(g) = \int_{\mathbb{R}^2} \int_{O(2, \mathbb{R})} f(T_x k) d\nu(k) dx
$$

$$
= \int_{\mathbb{R}^2} \sum_{\varepsilon = \pm 1} \int_{0}^{2\pi} f\left( T_x \begin{pmatrix}
\cos(\theta) & -\sin(\theta) \\
\varepsilon \sin(\theta) & \varepsilon \cos(\theta)
\end{pmatrix} \right) d\theta dx
$$

for every \( f \in C_c(\text{Iso}(\mathbb{R}^2)) \).

**Exercise 5.** (\( \text{Aut}(\mathbb{R}^n, +) \cong \text{GL}(n, \mathbb{R}) \)):  

For a topological group \( G \), we denote by \( \text{Aut}(G) \) the group of bijective, continuous homomorphisms of \( G \) with continuous inverse. Consider the locally compact Hausdorff group \( G = (\mathbb{R}^n, +) \) where \( n \in \mathbb{N}_0 \).

a) Show that \( \text{Aut}(G) \), i.e. the group of bijective homomorphisms which are homeomorphisms as well, is given by \( \text{GL}_n(\mathbb{R}) \).

**Solution:** Let \( \varphi \in \text{Aut}(\mathbb{R}^n) \), then \( \varphi \) is in particular additive and thus \( \varphi(nv) = n\varphi(v) \) for all \( v \in \mathbb{R}^n \), for all \( n \in \mathbb{Z} \). Let \( m \in \mathbb{Z}, n \in \mathbb{N} \) and \( q = \frac{m}{n} \in \mathbb{Q} \), then

$$
n\varphi(qv) = \varphi(nqv) = \varphi(mv) = m\varphi(v) \implies \varphi(q)\varphi(v) = q\varphi(v)
$$

and \( \varphi \) is \( \mathbb{Q} \)-linear. \( \mathbb{R} \)-linearity follows from continuity of \( \varphi \) and thus \( \varphi \in \text{End}_{\mathbb{R}}(\mathbb{R}^n) \). As \( \varphi \) is invertible, any choice of basis realizes \( \varphi \) as an element in \( \text{GL}_n(\mathbb{R}) \). It is clear that for such a choice of a basis, any \( g \in \text{GL}_n(\mathbb{R}) \) defines an element in \( \text{Aut}(\mathbb{R}^n) \) and that the correspondence is 1-1 and obeys the various
b) Show that \( \text{mod} : \text{Aut}(G) \to \mathbb{R}_{>0} \) is given by \( \alpha \mapsto |\det \alpha|^{-1} \).

Solution:

The \( n \)-dimensional Lebesgue measure \( \lambda_n \) on \( \mathbb{R}^n \) clearly is a Haar measure for \( \mathbb{R}^n \): it is translation invariant and
\[
\lambda_n( B_r(v)) = \frac{(\sqrt{\pi}r)^n}{\Gamma(\frac{n}{2} + 1)} \in (0, \infty) \quad (r > 0, v \in \mathbb{R}^n),
\]
showing that it is positive on open and finite on compact subsets of \( \mathbb{R}^n \). Let \( f \in C_c(\mathbb{R}^n), g \in \text{GL}_n(\mathbb{R}) \), then
\[
\int_{\mathbb{R}^n} f(gv) \, d\lambda_n(v) = \frac{1}{|\det g|} \int_{\mathbb{R}^n} f(gv)|\det g| \, d\lambda_n(v)
= |\det g|^{-1} \int_{\mathbb{R}^n} f(v) \, d\lambda_n(v).
\]
As any Borel measure on \( \mathbb{R}^n \) is uniquely determined by its values on \( C_c(\mathbb{R}^n) \), it follows \( g_*\lambda_n = |\det g|^{-1} \lambda_n \) and hence the claim.

c) Prove that there exists a discontinuous, bijective homomorphism from the additive group \((\mathbb{R},+)\) to itself.

Solution: Using Zorn’s lemma, construct a \( \mathbb{Q} \)-basis of \( \mathbb{R} \) containing 1. Denote this basis by \( \{x_i; i \in I\} \) for any infinite index set \( I \) containing 0 such that \( x_0 = 1 \) (\( I \) is infinite as otherwise \( \mathbb{R} \) would be algebraic over \( \mathbb{Q} \)). Fix \( i, j \in I \setminus \{0\} \) such that \( i \neq j \) and define a linear map \( \varphi : \mathbb{R} \to \mathbb{R} \) by \( \mathbb{Q} \)-linear extension of
\[
\forall k \in I : \varphi(x_k) = \begin{cases} x_j & \text{if } k = i, \\ x_i & \text{if } k = j, \\ x_k & \text{else}. \end{cases}
\]

Let \( (q_n)_{n \in \mathbb{N}} \in \mathbb{Q}^N \) Cauchy such that \( \lim_{n \to \infty} q_n = x_i \), then
\[
\lim_{n \to \infty} \varphi(q_n) = \lim_{n \to \infty} q_n = x_i \neq x_j = \varphi(x_i) = \varphi(\lim_{n \to \infty} q_n).
\]

Exercise 6.(Iterated Quotient Measures):

Let \( G \) be a locally compact Hausdorff group. Show that if \( H_1 \leq H_2 \leq G \) are closed
subgroups and $H_1, H_2, G$ are all unimodular then there exist invariant measures $dx, dy, dz$ on $G/H_1, G/H_2$ and $H_2/H_1$ respectively such that

$$\int_{G/H_1} f(x) dx = \int_{G/H_2} \left( \int_{H_2/H_1} f(yz) dz \right) dy$$

for all $f \in C_c(G/H_1)$.

**Solution:** We will use the existence of quotient measures here extensively. Note that $H_1, H_2$ and $G$ are unimodular such that the necessary and sufficient condition for the existence of quotient measures is always met.

Let $dg$ be a Haar measure on $G$ and $dh_1$ a Haar measure on $H_1$. We have $\Delta_G|H_1 \equiv 1 \equiv \Delta_{H_1}$ such that there is a left-invariant measure $dx$ on $G/H_1$ satisfying

$$\int_G F(g) dg = \int_{G/H_1} \int_{H_1} F(xh_1) dh_1 dx,$$

for every $F \in C_c(G)$.

Let $dh_2$ be a Haar measure on $H_2$. We have $\Delta_{H_2}|H_1 \equiv 1 \equiv \Delta_{H_1}$ such that there is a left-invariant measure $dz$ on $H_2/H_1$ satisfying

$$\int_{H_2} F(h_2) dh_2 = \int_{H_2/H_1} \int_{H_1} F(zh_1) dh_1 dz,$$

for every $F \in C_c(H_2)$.

Finally, we have $\Delta_G|H_2 \equiv 1 \equiv \Delta_{H_2}$ such that there is a left-invariant measure $dy$ on $G/H_2$ satisfying

$$\int_G F(g) dg = \int_{G/H_2} \int_{H_2} F(yh_2) dh_2 dy,$$

for every $F \in C_c(G)$.

We claim that these measures satisfy the hypothesis.

Let $f \in C_c(G/H_1)$. By a lemma from the lecture we may find an $F \in C_c(G)$ such that

$$f(gH_1) = \int_{H_1} F(gh_1) dh_1.$$
We compute
\[
\int_{G/H} f(x)dx = \int_{G/H_1} \int_{H_1} F(xh_1)dh_1dx \\
= \int_{G/H_2} \int_{H_2} F(yh_2)dydh_2y \\
= \int_{G/H_2} \int_{H_2/H_1} \int_{H_1} F(yzh_1)dh_1dzdy \\
= \int_{G/H_2} \int_{H_2/H_1} \int_{H_1} F(yz)dzdy.
\]

\[
\square
\]

**Exercise 7. (No SL\(_2(\mathbb{R})\)-invariant Measure on SL\(_2(\mathbb{R})/P\)):**

Let \( G = SL_2(\mathbb{R}) \) and \( P \) be the subgroup of upper triangular matrices. Show directly that there is no (non-trivial) finite \( G \)-invariant measure on \( G/P \).

**Hint:** Identify \( G/P \cong S^1 \cong \mathbb{R} \cup \{\infty\} \) with the unit circle and consider a rotation
\[
\begin{pmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{pmatrix}
\]
and a translation
\[
\begin{pmatrix}
1 & t \\
0 & 1
\end{pmatrix}.
\]

**Solution:** Recall that \( G = SL_2(\mathbb{R}) \) acts on the upper half plane \( \mathbb{H} = \{z \in \mathbb{C} | \text{Im}z > 0\} \subset \hat{\mathbb{C}} \) and its boundary \( \partial \mathbb{H} = \mathbb{R} \cup \{\infty\} \subset \hat{\mathbb{C}} \) via Möbius transformations
\[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}.z = \frac{az+b}{cz+d}.
\]

Note that \( SL_2(\mathbb{R}) \) acts transitively on \( \partial \mathbb{H} \) and the stabilizer of \( \infty \) is the subgroup of upper triangular matrices \( P \). We may therefore identify \( G/P \cong \mathbb{R} \cup \{\infty\} \).

Suppose there is a finite \( G \)-invariant measure \( m \) on \( G/P \cong \mathbb{R} \cup \{\infty\} \). Consider the restriction \( \mu = m|_\mathbb{R} \) of this measure to the real line. Observe that \( G \) acts on the real line via translations
\[
\begin{pmatrix}
1 & t \\
0 & 1
\end{pmatrix}.\xi = \xi + t, \quad \xi, t \in \mathbb{R},
\]
such that \( \mu \) is in particular a translation invariant measure on \( \mathbb{R} \), i.e. \( \mu \) is a Haar measure on \( \mathbb{R} \). By uniqueness of Haar measures \( \mu \) must be a multiple of the Lebesgue measure on \( \mathbb{R} \). Since \( m \) is finite \( \mu \) is the zero measure. That means that \( m \) is a positive multiple of the dirac measure at \( \infty \), i.e. \( m = \lambda \cdot \delta_\infty \) for some \( \lambda > 0 \). Now
consider the rotation
\[ i(z) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot z = -\frac{1}{z} \]
that sends \( \infty \) to 0. By \( G \)-invariance we must have
\[ \lambda \cdot \delta_\infty = i_*(\lambda \cdot \delta_\infty) = \lambda \cdot \delta_0 \]
such that \( \lambda = 0 \); in contradiction to our assumption. \( \square \)