Prof. Dr. A. Iozzi | Introduction to Lie Groups | Autumn 2020
Y. Krifka

# Solution Exercise Sheet 2

## Exercise 1.(Identity Neighborhoods Generate Connected Groups):

Let $G$ be a connected topologcial group, $U \subset G$ an open neighborhood of the identity and $U^n := \{g_1 \cdots g_n | g_1, \dots, g_n \in U\}$. Show that $G = \bigcup_{n=1}^{\infty} U^n$.

<u>Hint:</u> You may assume that $g^{-1} \in U$ for every $g \in U$. Why?

**Solution:** By replacing $U$ with $U \cap U^{-1}$ if necessary we may assume that $U$ is a symmetric neighborhood $U = U^{-1}$ of the identity $e \in G$.

Observe that $H = \bigcup_{n=1}^{\infty} U^n$ is a group. Indeed, for every $g_1 \cdots g_n, h_1 \cdots h_m \in H$ also

$$(g_1 \cdots g_n) \cdot (h_1 \cdots h_m)^{-1} = g_1 \cdots g_n \cdot h_m^{-1} \cdots h_1^{-1} \in U^{n+m} \subset H.$$

Further, $U$ is open and therefore also every

$$U^n = \bigcup_{g \in U^{m-1}} U \cdot g \subset G$$

is open as the union of open sets. Recall that right translation by group elements is a homeomorphism.

Hence, also $H = \bigcup_{n=1}^{\infty} U^n$ is open, i.e. $H \subset G$ is an open subgroup. In the lecture we have learned that open subgroups are always closed. Because $G$ is connected we have therefore $H = \emptyset$ or $H = G$. Since $H$ is non-empty the assertion follows.

$\square$

## Exercise 2.(Transitive Group Actions):

Let $G$ be a topological group, $X$ a topological space and $\mu : G \times X \to X$ a continuous transitive group action, i.e. for any two $x, y \in X$ there is $g \in G$ such that $\mu(g, x) = g \cdot x = y$.

a) Show that if $G$ is compact then $X$ is compact.

b) Show that if $G$ is connected then $X$ is connected.

**Solution:** Let $x_0 \in X$ and consider the map

$$\begin{aligned} \varphi : G &\to X, \\ g &\mapsto \mu(g, x_0). \end{aligned}$$

Because $\mu$ is a continuous action the map $\varphi$ is continuous too. Further the action

$\mu$ is transitive, i.e. for every $y \in X$ there is a $g \in G$ such that $\mu(g, x_0) = y$. In other words, $\varphi$ is surjective.

Part a) follows from the fact that $X = \varphi(G)$ is compact as the image of a compact group.

Part b) follows from the fact that continuous maps send connected components to connected components and again that $\varphi(G) = X$. $\square$

**Exercise 3.(Examples of Haar Measures):**

a) Let us consider the *three-dimensional Heisenberg group* $H = \mathbb{R} \rtimes_\eta \mathbb{R}^2$, where $\eta : \mathbb{R} \to \mathrm{Aut}(\mathbb{R}^2)$ is defined by

$$\eta(x)\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} y \\ z + xy \end{pmatrix},$$

for all $x, y, z \in \mathbb{R}$. Thus the group operation is given by

$$(x_1, y_1, z_1) * (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2 + x_1 y_2)$$

and it is easy to see that it can be identified with the matrix group

$$H \cong \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}$$

Verify that the Lebesgue measure is the Haar measure of $\mathbb{R} \rtimes_\eta \mathbb{R}^2$ and that the group is unimodular.

**Solution:** Denote by $\mu$ the measure on $H$ induced by the Lebesgue measure on $\mathbb{R}^3$. In order to show that $\mu$ is unimodular we need to see that

$$\mu(\lambda(h)f) = \mu(f) = \mu(\rho(h)f)$$

for every $f \in C_c(H)$, $h \in H$.

Let $h_1 = (x_1, y_1, z_1) \in H$ and $f \in C_c(H)$. We compute

$$
\begin{aligned}
&\int (\lambda(h_1^{-1})f)(x_2,y_2,z_2)dx_2dy_2dz_2 \\
&= \int f(x_1+x_2,y_1+y_2,z_1+z_2+x_1y_2)dx_2dy_2dz_2 \\
&\overset{\text{Fubini}}{=} \int f(x_1+x_2,y_1+y_2,z_2+(z_1+x_1y_2))dz_2dx_2dy_2 \\
&\overset{\text{transl. inv.}}{=} \int f(x_1+x_2,y_1+y_2,z_2)dz_2dx_2dy_2 \\
&\overset{\text{F. \& t.i.}}{=} \int f(x_1,y_1+y_2,z_2)dx_2dy_2dz_2 \\
&\overset{\text{F. \& t.i.}}{=} \int f(x_1,y_2,z_2)dx_2dy_2dz_2.
\end{aligned}
$$

This shows left-invariance.

$$
\begin{aligned}
&\int (\rho(h_1)f)(x_2,y_2,z_2)dx_2dy_2dz_2 \\
&= \int f(x_2+x_1,y_2+y_1,z_2+z_1+x_2y_1)dx_2dy_2dz_2 \\
&\overset{\text{Fubini}}{=} \int f(x_1+x_2,y_1+y_2,z_2+(z_1+x_2y_1))dz_2dx_2dy_2 \\
&\overset{\text{transl. inv.}}{=} \int f(x_1+x_2,y_1+y_2,z_2)dz_2dx_2dy_2 \\
&\overset{\text{F. \& t.i.}}{=} \int f(x_1,y_1+y_2,z_2)dx_2dy_2dz_2 \\
&\overset{\text{F. \& t.i.}}{=} \int f(x_1,y_2,z_2)dx_2dy_2dz_2.
\end{aligned}
$$

This shows right-invariance. Therefore $\mu$ is a left- and right-invariant Haar measure on $H$ and $H$ is unimodular.

□

b) Let

$$P = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a,b \in \mathbb{R}, a \neq 0 \right\}.$$

Show that $\frac{da}{a^2}\,db$ is the left Haar measure and $da\,db$ is the right Haar measure. In particular, $P$ is *not* unimodular.

**Solution:** Let $\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \in P$ and $f \in C_c(P)$. We compute

$$\begin{aligned}
&\int \left( \lambda\left( \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}^{-1} \right) f \right) \begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix} \frac{dx}{x^2} dy \\
&= \int f\left( \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix} \right) \frac{dx}{x^2} dy \\
&= \int f\begin{pmatrix} ax & ay + bx^{-1} \\ 0 & a^{-1}x^{-1} \end{pmatrix} a^2 \frac{dx}{(ax)^2} dy = \dots
\end{aligned}$$

we change coordinates to $\bar{x} = ax, \bar{y} = ay$ which has Jacobi determinant $a^2$

$$\begin{aligned}
\dots &= \int f\begin{pmatrix} \bar{x} & \bar{y} + ab\bar{x}^{-1} \\ 0 & \bar{x}^{-1} \end{pmatrix} \frac{d\bar{x}}{\bar{x}^2} d\bar{y} \\
&= \int f\begin{pmatrix} \bar{x} & \bar{y} + ab\bar{x}^{-1} \\ 0 & \bar{x}^{-1} \end{pmatrix} d\bar{y} \frac{d\bar{x}}{\bar{x}^2} \\
&= \int f\begin{pmatrix} \bar{x} & \bar{y} \\ 0 & \bar{x}^{-1} \end{pmatrix} \frac{d\bar{x}}{\bar{x}^2} d\bar{y}.
\end{aligned}$$

This shows left-invariance for the measure $\frac{dx}{x^2}dy$ as claimed.

We will now see that $da\,db$ is right-invariant:

$$\begin{aligned}
&\int \left( \rho\left( \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \right) f \right) \begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix} dxdy \\
&= \int f\left( \begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix} \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \right) dxdy \\
&= \int f\left( \begin{pmatrix} ax & bx + a^{-1}y \\ 0 & a^{-1}x^{-1} \end{pmatrix} \right) dxdy = \dots
\end{aligned}$$

we change coordinates to $\bar{x} = ax, \bar{y} = a^{-1}y$ which has Jacobi determinant 1

$$\begin{aligned}
\dots &= \int f\left( \begin{pmatrix} \bar{x} & ba^{-1}\bar{x} + \bar{y} \\ 0 & \bar{x}^{-1} \end{pmatrix} \right) d\bar{x}d\bar{y} \\
&\overset{\text{F \& t.i}}{=} \int f\left( \begin{pmatrix} \bar{x} & \bar{y} \\ 0 & \bar{x}^{-1} \end{pmatrix} \right) d\bar{x}d\bar{y}
\end{aligned}$$

This shows right-invariance. Since both measures clearly do not coincide $P$ is *not* unimodular.

□

c) Let $G := \mathrm{GL}_n(\mathbb{R}) \subseteq \mathbb{R}^{n^2}$ denote the group of invertible matrices over $\mathbb{R}$. Let $\lambda_{n^2}$

denote the Lebesgue measure on $\mathbb{R}^{n^2}$. Prove that

$$\mathrm{d}m(x) := |\det x|^{-n}\,\mathrm{d}\lambda_{n^2}(x)$$

defines a bi-invariant (i.e. left- and right-invariant) Haar measure on $G$.

**Solution:** As $\mathrm{GL}_n(\mathbb{R}) = \det^{-1}\big(\mathbb{R}\setminus\{0\}\big)$ is open in $\mathbb{R}^{n^2}$, $\lambda_{n^2}\big|_{\mathrm{GL}_n(\mathbb{R})}$ assigns non-zero measure to non-empty open and finite measure to compact subsets of $\mathrm{GL}_n(\mathbb{R})$ (if $K\subseteq \mathrm{GL}_n(\mathbb{R})$ is compact in $\mathrm{GL}_n(\mathbb{R})$ and $\mathcal{U}$ an open cover of $K$ in $\mathbb{R}^{n^2}$, then $\mathcal{U}\cap\mathrm{GL}_n(\mathbb{R}) := \{U\cap \mathrm{GL}_n(\mathbb{R}); U\in\mathcal{U}\}$ is an open cover of $K$ in $\mathrm{GL}_n(\mathbb{R})$, thus it admits a finite subcover and hence so does $\mathcal{U}$). As det is continuous and does not vanish on $\mathrm{GL}_n(\mathbb{R})$, the above also holds for $dm(g) := |\det g|^{-n}\, d\lambda_{n^2}(g)$. It remains to show that $m$ is invariant. To this end we note that for $g\in\mathrm{GL}_n(\mathbb{R})$, if $g=(g_1,\dots,g_n)$ and $h\in\mathrm{GL}_n(\mathbb{R})$, then

$$hg = (hg_1,\dots,hg_2) \quad (g\in \mathrm{Mat}_n(\mathbb{R})),$$

so that the left-action of $h$ on $\mathrm{GL}_n(\mathbb{R})$ can be viewed as a restriction of a diagonal matrix $\mathrm{diag}(h,\dots,h)\in\mathbb{R}^{n^2\times n^2}$ acting on a subset of $\mathbb{R}^{n^2}$. Let $f\in C_c\big(\mathrm{GL}_n(\mathbb{R})\big)$, then

$$\begin{aligned}
&\int_{\mathbb{R}^{n^2}} \mathbb{1}_{\mathrm{GL}_n(\mathbb{R})}(g) f(hg)\,|\det g|^{-n}\, d\lambda_{n^2}(g)\\
&\qquad = \int_{\mathbb{R}^{n^2}} \mathbb{1}_{h\mathrm{GL}_n(\mathbb{R})}(hg) f(hg)\,|\det hg|^{-n}\,|\det h|^{n}\, d\lambda_{n^2}(g)\\
&\qquad = \int_{\mathbb{R}^{n^2}} \mathbb{1}_{\mathrm{GL}_n(\mathbb{R})}(hg) f(hg)\,|\det hg|^{-n}\,|\det h|^{n}\, d\lambda_{n^2}(g)\\
&\qquad = \int_{\mathbb{R}^{n^2}} \mathbb{1}_{\mathrm{GL}_n(\mathbb{R})}(g) f(g)\,|\det g|^{-n}\, d\lambda_{n^2}(g),
\end{aligned}$$

where in the end we used the substitution formula for the map $\mathrm{diag}(h,\dots,h)$. This proves that $m$ is a left Haar measure on $\mathrm{GL}_n(\mathbb{R})$. The measure is also right-invariant, because the map

$$g\mapsto \begin{pmatrix} g_1 h\\ \vdots \\ g_n h\end{pmatrix}$$

does also have Jacobian $|\det h|^n$ (for example because $gh = (h^t g^t)^t$ and the Jacobian of transposition – being an idempotent map – is equal to 1). Thus $\mathrm{GL}_n(\mathbb{R})$ is unimodular. □

d) Let $G=\mathrm{SL}_n(\mathbb{R})$ denote the group of matrices of determinant 1 in $\mathbb{R}^{n\times n}$. For a Borel subset $B\subseteq \mathrm{SL}_n(\mathbb{R})$ define

$$m(B) := \lambda_{n^2}\Big(\{tg; g\in B, t\in[0,1]\}\Big).$$

Show that $m$ is a well-defined bi-invariant Haar measure on $\mathrm{SL}_n(\mathbb{R})$.

**Solution:** Let us first check that for any Borel subset $B \subseteq \mathrm{SL}_n(\mathbb{R})$ the cone

$$\mathcal{C}(B) = \{tb : b \in B, t \in [0,1]\}$$

is a Borel subset of $\mathbb{R}^{n^2}$. To this end we note first that

$$\mathcal{C}(B) = \mathcal{C}'(B) \cup \{0\},$$

where

$$\mathcal{C}'(B) = \{tb : b \in B, t \in (0,1]\}.$$

It clearly suffices to show that $\mathcal{C}'(B)$ is Borel. To this end let

$$\mathrm{GL}_n^1(\mathbb{R}) = \{g \in \mathrm{GL}_n(\mathbb{R}); |\det g| = 1\}.$$

Note that $\mathrm{GL}_n^1(\mathbb{R}) \cong \mathrm{SL}_n(\mathbb{R}) \rtimes C_2$, where $C_2$ is the group with two elements. As $\mathrm{GL}_n^1(\mathbb{R})$ is homeomorphic to a disjoint union of two copies of $\mathrm{SL}_n(\mathbb{R})$, $B$ is Borel in $\mathrm{GL}_n^1(\mathbb{R})$. Define

$$\Psi : \mathrm{GL}_n(\mathbb{R}) \to \mathrm{GL}_n^1(\mathbb{R}), \quad g \mapsto \frac{1}{\sqrt[n]{|\det g|}} g.$$

This is a Borel map and therefore

$$\mathcal{C}'(B) = \Psi^{-1}(B) \cap \det{}^{-1}(0,1]$$

is measurable.

The final claim now follows immediately from the argument in part c), which realizes the action of an element $g \in \mathrm{SL}_n(\mathbb{R})$ on $\mathbb{R}^{n^2}$ as a diagonal action of $n$ copies of $g$, together with the fact that $\Phi_* \lambda_{n^2} = |\det \Phi| \lambda_{n^2}$ for linear $\Phi$, $\det g = 1$, $\mathcal{C}(gB) = g\mathcal{C}(B)$ and $\mathcal{C}(Bg) = \mathcal{C}(B)g$ for all $g \in \mathrm{SL}_n(\mathbb{R})$ and $B \subseteq \mathrm{SL}_n(\mathbb{R})$ Borel. □

e) Let $G$ denote the $ax+b$ group defined as

$$G = \left\{ \begin{pmatrix} a & b \\ & 1 \end{pmatrix}; a \in \mathbb{R}^\times, b \in \mathbb{R} \right\}$$

Note that every element in $G$ can be written in a unique fashion as a product of the form:

$$\begin{pmatrix} a & b \\ & 1 \end{pmatrix} = \begin{pmatrix} \alpha & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \beta \\ & 1 \end{pmatrix}$$

where $\alpha \in \mathbb{R}^\times$ and $\beta \in \mathbb{R}$, which yields a coordinate system $\mathbb{R}^\times \times \mathbb{R} \leftrightarrow G$. Prove that

$$\mathrm{d}m(\alpha,\beta) = \frac{1}{|\alpha|}\,\mathrm{d}\alpha\,\mathrm{d}\beta$$

defines a left Haar measure on $G$. Calculate $\Delta_G(\alpha,\beta)$ for $\alpha \in \mathbb{R}^\times$ and $\beta \in \mathbb{R}$.

**Solution:** We use the coordinate system $\varphi : \mathrm{Aff}_1(\mathbb{R}) \ni (a,b) \mapsto (a,a^{-1}b) \in \mathbb{R}^\times \times \mathbb{R}$. On $\mathbb{R}^\times \times \mathbb{R}$ we define the measure $d\nu(\alpha,\beta) := \frac{1}{|\alpha|}\,d\alpha\,d\beta$ and we claim that $(\varphi^{-1})_*\nu$ is a left-Haar measure on $\mathrm{Aff}_1(\mathbb{R})$.

Let $f \in C_c\big(\mathrm{Aff}_1(\mathbb{R})\big)$ and denote $\psi(\alpha) := x\alpha$, then for left-translation – indicated by subscript – follows

$$\begin{aligned}
(\varphi^{-1})_*\nu\Big(f_{\left(\begin{smallmatrix} x & y \\ & 1\end{smallmatrix}\right)}\Big) &= \int_{\mathbb{R}^\times}\left(\int_{\mathbb{R}} \frac{f_{\left(\begin{smallmatrix} x & y \\ & 1\end{smallmatrix}\right)} \circ \varphi^{-1}(\alpha,\beta)}{|\alpha|}\,d\beta\right) d\alpha \\
&= \int_{\mathbb{R}^\times}\left(\int_{\mathbb{R}} \frac{f \circ \varphi^{-1}\big(x\alpha, \beta + (x\alpha)^{-1}y\big)}{|\alpha|}\,d\beta\right) d\alpha \\
\text{(trans. inv.)} \qquad &= \int_{\mathbb{R}^\times}\left(\int_{\mathbb{R}} \frac{f \circ \varphi^{-1}(x\alpha,\beta)}{|\alpha|}\,d\beta\right) d\alpha \\
(\psi'(x) = x) \qquad &= \int_{\mathbb{R}}\left(\int_{\mathbb{R}^\times} \frac{f \circ \varphi^{-1}\big(\psi(\alpha),\beta\big)}{\big|\psi(\alpha)\big|}\,\big|\psi'(\alpha)\big|\,d\alpha\right) d\beta \\
(\psi(\mathbb{R}^\times) = \mathbb{R}^\times) \qquad &= \int_{\mathbb{R}}\left(\int_{\mathbb{R}^\times} \frac{f \circ \varphi^{-1}\big(\alpha,\beta\big)}{|\alpha|}\,d\alpha\right) d\beta = (\varphi^{-1})_*\nu(f)
\end{aligned}$$

and thus we have indeed found a left Haar measure. For right translation – indicated by superscript – follows

$$\begin{aligned}
(\varphi^{-1})_*\nu\Big(f^{\left(\begin{smallmatrix} x & y \\ & 1\end{smallmatrix}\right)}\Big) &= \int_{\mathbb{R}^\times}\left(\int_{\mathbb{R}} \frac{f^{\left(\begin{smallmatrix} x & y \\ & 1\end{smallmatrix}\right)} \circ \varphi^{-1}(\alpha,\beta)}{|\alpha|}\,d\beta\right) d\alpha \\
&= \int_{\mathbb{R}^\times}\left(\int_{\mathbb{R}} \frac{f \circ \varphi^{-1}\big(x\alpha, x^{-1}\beta + x^{-1}y\big)}{|\alpha|}\,d\beta\right) d\alpha \\
\text{(trans. inv.)} \qquad &= \int_{\mathbb{R}^\times}\left(\int_{\mathbb{R}} \frac{f \circ \varphi^{-1}\big(x\alpha, x^{-1}\beta\big)}{|\alpha|}\,d\beta\right) d\alpha \\
\text{(subst. } \beta \mapsto x\beta) \qquad &= \int_{\mathbb{R}^\times}|x|\left(\int_{\mathbb{R}} \frac{f \circ \varphi^{-1}\big(x\alpha, \beta\big)}{|\alpha|}\,d\beta\right) d\alpha \\
\text{(as above)} \qquad &= |x|\,(\varphi^{-1})_*\nu(f).
\end{aligned}$$

Hence $\Delta_{\mathrm{Aff}_1(\mathbb{R})}\left(\begin{smallmatrix} x & y \\ & 1\end{smallmatrix}\right) = |x|^{-1}$. □

**Exercise 4.(Haar Measure and Transitive Actions):**

Let $G$ be a locally compact Hausdorff group and let $X$ be a topological space.

Suppose that $G$ acts on $X$ continuously and transitively. Let $o \in X$, and denote $\pi\colon G \to X, g \mapsto g \cdot o$. Further, let

$$H := \mathrm{Stab}(o) = \{h \in G \mid h \cdot o = o\}$$

be the stabilizer of $o$.

Suppose there is a continuous section $\sigma\colon X \to G$ of $\pi$, i.e. $\pi \circ \sigma = \mathrm{Id}_X$.

a) Show that $\psi\colon X \times H \to G, (x,h) \mapsto \sigma(x)h$ is a homeomorphism.

Hint: Find a continuous inverse!

**Solution:** We define $\varphi\colon G \to X \times H$ via

$$\varphi(g) := (\pi(g), \sigma(\pi(g))^{-1} g)$$

for all $g \in G$.

Note that

$$\sigma(\pi(g)) \cdot o = \pi(\sigma(\pi(g)) = \pi(g) = g \cdot o,$$

whence $\sigma(\pi(g))^{-1} g \cdot o = o$ and $\sigma(\pi(g))^{-1} g \in H = \mathrm{Stab}(o)$. This shows that $\varphi$ is well-defined. Moreover, $\varphi$ is continuous as a composition of continuous functions.

We will now show that $\varphi$ is the inverse of $\psi$, i.e. $\psi \circ \varphi = \mathrm{Id}_G$ and $\varphi \circ \psi = \mathrm{Id}_{X \times H}$.

Let $g \in G$. We compute:

$$\begin{aligned}
\psi(\varphi(g)) &= \psi(\pi(g), \sigma(\pi(g))^{-1} g) \\
&= \sigma(\pi(g))\sigma(\pi(g))^{-1} g = g.
\end{aligned}$$

Let $x \in X, h \in H$. We compute:

$$\begin{aligned}
\varphi(\psi(x,h)) &= \varphi(\sigma(x)h) \\
&= (\pi(\sigma(x)h), \sigma(\pi(\sigma(x)h))^{-1} \sigma(x)h) \\
&= (\sigma(x)h \cdot o, \sigma(\sigma(x)h \cdot o)^{-1} \sigma(x)h) \\
&= (\sigma(x) \cdot o, \sigma(\sigma(x) \cdot o)^{-1} \sigma(x)h) \\
&= (x, \sigma(x)^{-1} \sigma(x)h) \\
&= (x,h).
\end{aligned}$$

□

b) Suppose there is a (left) Haar measure $\nu$ on $H$ and suppose there is a left $G$-invariant Borel regular measure $\lambda$ on $X$.

Show that the push-forward measure $\psi_*(\lambda \otimes \nu)$ is a (left) Haar measure on $G$.

**Solution:** All we need to see is that the push-forward measure $\mu = \psi_*(\lambda \otimes \nu)$ is left $G$-invariant.

Let $f \in C_c(G)$ and $g_0 \in G$. We compute:

$$\begin{aligned}
\int_G f(g_0 g)\, d\mu(g) &= \int_{X\times H} f(g_0\psi(x,h))\, d(\lambda\otimes\nu)(x,h) \\
\text{(Fubini)} \quad &= \int_X \int_H f(g_0\sigma(x)h)\, d\nu(h)\, d\lambda(x) \\
&= \int_X \int_H f(\sigma(g_0\cdot x)\underbrace{\sigma(g_0\cdot x)^{-1} g_0 \sigma(x) h}_{\in H})\, d\nu(h)\, d\lambda(x) \\
\text{(left invariance of } \nu) \quad &= \int_X \int_H f(\sigma(g_0\cdot x)h)\, d\nu(h)\, d\lambda(x) \\
\text{(left } G\text{-invariance of } \lambda) \quad &= \int_X \int_H f(\sigma(x)h)\, d\nu(h)\, d\lambda(x) \\
&= \int_G f(g)\, d\mu(g)
\end{aligned}$$

□

c) Find a Haar measure on $\mathrm{Iso}(\mathbb{R}^2)$.

**Solution:** Note that $\mathrm{Iso}(\mathbb{R}^2)$ acts continuously and transitively on $\mathbb{R}^2$. Indeed, any translation $T_x\colon \mathbb{R}^2 \to \mathbb{R}^2, y \mapsto x+y$ ($x \in \mathbb{R}^2$) is a Euclidean isometry, that maps 0 to $x$.

In fact, this construction yields a continuous section $\sigma\colon \mathbb{R}^2 \to \mathrm{Iso}(\mathbb{R}^2), x \mapsto T_x$, and we can apply part b). Indeed, the Lebesgue measure $\lambda$ on $\mathbb{R}^2$ is $\mathrm{Iso}(\mathbb{R}^2)$-invariant, and it can be shown that the stabilizer of 0 is given by the orthogonal group $O(2,\mathbb{R})$. By part b), a Haar measure on $\mathrm{Iso}(\mathbb{R}^2)$ is given by the push-forward measure $\mu := \psi_*(\lambda \otimes \nu)$ where $\nu$ is a left Haar measure on $O(2,\mathbb{R})$.

We will apply part b) again to compute a Haar measure on $O(2,\mathbb{R})$ more explicitly. Observe that $O(2,\mathbb{R})$ acts transitively on the group with two elements $\{\pm 1\}$ via $k * \varepsilon := \det(k)\cdot\varepsilon$ for every $k \in O(2,\mathbb{R}), \varepsilon \in \{\pm 1\}$. We obtain a surjective map $p = \det\colon O(2,\mathbb{R}) \to \{\pm 1\}, k \mapsto \det(k)\cdot 1 = \det(k)$. A section $\tau\colon \{\pm 1\} \to O(2,\mathbb{R})$ of det is given by

$$\tau(\varepsilon) = \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix},$$

which is continuous because $\{\pm 1\}$ carries the discrete topology. The stabilizer of 1 is then $\det^{-1}(1) \cap O(2,\mathbb{R}) = \mathrm{SO}(2,\mathbb{R})$ and the usual Lebesgue measure on $[0,2\pi)$ pushes-forward to a left Haar measure $\xi = \varphi_*(\lambda|_{[0,2\pi)})$ on $\mathrm{SO}(2,\mathbb{R})$ via the map

$$\varphi\colon [0,2\pi) \longrightarrow \mathrm{SO}(2,\mathbb{R}),$$
$$\theta \longmapsto \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

Clearly, an invariant measure on $\{\pm 1\}$ is given by the counting measure. Therefore, a left Haar measure $\nu$ on $O(2,\mathbb{R})$ is given by

$$\begin{aligned}\int_{O(2,\mathbb{R})} f(k)\,d\nu(k) &= \sum_{\varepsilon=\pm 1}\int_0^{2\pi} f\left(\tau(\varepsilon)\cdot\varphi(\theta)\right)d\theta \\ &= \sum_{\varepsilon=\pm 1}\int_0^{2\pi} f\left(\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \varepsilon\sin(\theta) & \varepsilon\cos(\theta) \end{pmatrix}\right)d\theta\end{aligned}$$

for every $f \in C_c(O(2,\mathbb{R}))$.

Putting everything together we obtain

$$\begin{aligned}\int_{\mathrm{Iso}(\mathbb{R}^2)} f(g)\,d\mu(g) &= \int_{\mathbb{R}^2}\int_{O(2,\mathbb{R})} f(T_x k)\,d\nu(k)\,dx \\ &= \int_{\mathbb{R}^2}\sum_{\varepsilon=\pm 1}\int_0^{2\pi} f\left(T_x\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \varepsilon\sin(\theta) & \varepsilon\cos(\theta) \end{pmatrix}\right)d\theta\,dx\end{aligned}$$

for every $f \in C_c(\mathrm{Iso}(\mathbb{R}^2))$. □

**Exercise 5.($\mathrm{Aut}(\mathbb{R}^n,+) \cong \mathrm{GL}(n,\mathbb{R})$):**

For a topological group $G$, we denote by $\mathrm{Aut}(G)$ the group of bijective, continuous homomorphisms of $G$ with continuous inverse. Consider the locally compact Hausdorff group $G = (\mathbb{R}^n,+)$ where $n \in \mathbb{N}_0$.

a) Show that $\mathrm{Aut}(G)$, i.e. the group of bijective homomorphisms which are homeomorphisms as well, is given by $\mathrm{GL}_n(\mathbb{R})$.

**Solution:** Let $\varphi \in \mathrm{Aut}(\mathbb{R}^n)$, then $\varphi$ is in particular additive and thus $\varphi(nv) = n\varphi(v)$ for all $v \in \mathbb{R}^n$, for all $n \in \mathbb{Z}$. Let $m \in \mathbb{Z}$, $n \in \mathbb{N}$ and $q = \frac{m}{n} \in \mathbb{Q}$, then

$$n\varphi(qv) = \varphi(nqv) = \varphi(mv) = m\varphi(v) \implies \varphi(q)\varphi(v) = q\varphi(v)$$

and $\varphi$ is $\mathbb{Q}$-linear. $\mathbb{R}$-linearity follows from continuity of $\varphi$ and thus $\varphi \in \mathrm{End}_{\mathbb{R}}(\mathbb{R}^n)$. As $\varphi$ is invertible, any choice of basis realizes $\varphi$ as an element in $\mathrm{GL}_n(\mathbb{R})$. It is clear that for such a choice of a basis, any $g \in \mathrm{GL}_n(\mathbb{R})$ defines an element in $\mathrm{Aut}(\mathbb{R}^n)$ and that the correspondence is 1-1 and obeys the various group structures (on $\mathrm{Aut}(G)$ and $\mathrm{GL}_n(\mathbb{R})$).

$\square$

b) Show that $\mathrm{mod} : \mathrm{Aut}(G) \to \mathbb{R}_{>0}$ is given by $\alpha \mapsto |\det\alpha|^{-1}$.

**Solution:**

The $n$-dimensional Lebesgue measure $\lambda_n$ on $\mathbb{R}^n$ clearly is a Haar measure for $\mathbb{R}^n$: it is translation invariant and

$$\lambda_n\Big(B_r(v)\Big) = \frac{(\sqrt{\pi}r)^n}{\Gamma(\frac{n}{2}+1)} \in (0,\infty) \quad (r > 0, v \in \mathbb{R}^n),$$

showing that it is positive on open and finite on compact subsets of $\mathbb{R}^n$. Let $f \in C_c(\mathbb{R}^n)$, $g \in \mathrm{GL}_n(\mathbb{R})$, then

$$\begin{aligned}\int_{\mathbb{R}^n} f(gv)\, d\lambda_n(v) =& \frac{1}{|\det g|} \int_{\mathbb{R}^n} f(gv)\, |\det g|\, d\lambda_n(v)\\ =& |\det g|^{-1} \int_{\mathbb{R}^n} f(v)\, d\lambda_n(v).\end{aligned}$$

As any Borel measure on $\mathbb{R}^n$ is uniquely determined by its values on $C_c(\mathbb{R}^n)$, it follows $g_*\lambda_n = |\det g|^{-1}\, \lambda_n$ and hence the claim.

$\square$

c) Prove that there exists a discontinuous, bijective homomorphism from the additive group $(\mathbb{R},+)$ to itself.

**Solution:** Using Zorn's lemma, construct a $\mathbb{Q}$-basis of $\mathbb{R}$ containing 1. Denote this basis by $\{x_i; i \in I\}$ for any infinite index set $I$ containing 0 such that $x_0 = 1$ ($I$ is infinite as otherwise $\mathbb{R}$ would be algebraic over $\mathbb{Q}$). Fix $i, j \in I \setminus \{0\}$ such that $i \neq j$ and define a linear map $\varphi : \mathbb{R} \to \mathbb{R}$ by $\mathbb{Q}$-linear extension of

$$\forall k \in I : \varphi(x_k) = \begin{cases} x_j & \text{if } k = i,\\ x_i & \text{if } k = j,\\ x_k & \text{else.}\end{cases}$$

Let $(q_n)_{n\in\mathbb{N}} \in \mathbb{Q}^{\mathbb{N}}$ Cauchy such that $\lim_{n\to\infty} q_n = x_i$, then

$$\lim_{n\to\infty} \varphi(q_n) = \lim_{n\to\infty} q_n = x_i \neq x_j = \varphi(x_i) = \varphi(\lim_{n\to\infty} q_n).$$

$\square$

**Exercise 6.(Iterated Quotient Measures):**

Let $G$ be a locally compact Hausdorff group. Show that if $H_1 \leq H_2 \leq G$ are closed subgroups and $H_1, H_2, G$ are all unimodular then there exist invariant measures $dx$, $dy$, $dz$ on $G/H_1, G/H_2$ and $H_2/H_1$ respectively such that

$$\int_{G/H_1} f(x)dx = \int_{G/H_2} \left( \int_{H_2/H_1} f(yz)dz \right) dy$$

for all $f \in C_c(G/H_1)$.

**Solution:** We will use the existence of quotient measures here extensively. Note that $H_1, H_2$ and $G$ are unimodular such that the necessary and sufficient condition for the existence of quotient measures is always met.

Let $dg$ be a Haar measure on $G$ and $dh_1$ a Haar measure on $H_1$. We have $\Delta_G|_{H_1} \equiv 1 \equiv \Delta_{H_1}$ such that there is a left-invariant measure $dx$ on $G/H_1$ satisfying

$$\int_G F(g)dg = \int_{G/H_1} \int_{H_1} F(xh_1)dh_1 dx,$$

for every $F \in C_c(G)$.

Let $dh_2$ be a Haar measure on $H_2$. We have $\Delta_{H_2}|_{H_1} \equiv 1 \equiv \Delta_{H_1}$ such that there is a left-invariant measure $dz$ on $H_2/H_1$ satisfying

$$\int_{H_2} F(h_2)dh_2 = \int_{H_2/H_1} \int_{H_1} F(zh_1)dh_1 dz,$$

for every $F \in C_c(H_2)$.

Finally, we have $\Delta_G|_{H_2} \equiv 1 \equiv \Delta_{H_2}$ such that there is a left-invariant measure $dy$ on $G/H_2$ satisfying

$$\int_G F(g)dg = \int_{G/H_2} \int_{H_2} F(yh_2)dh_2 dy,$$

for every $F \in C_c(G)$.

We claim that these measures satisfy the hypothesis.

Let $f \in C_c(G/H_1)$. By a lemma from the lecture we may find an $F \in C_c(G)$ such that

$$f(gH_1) = \int_{H_1} F(gh_1)dh_1.$$

We compute

$$
\begin{aligned}
\int_{G/H_1} f(x)dx &= \int_{G/H_1}\int_{H_1} F(xh_1)dh_1 dx \\
&= \int_G F(g)dg \\
&= \int_{G/H_2}\int_{H_2} F(yh_2)dh_2 dy \\
&= \int_{G/H_2}\int_{H_2/H_1}\int_{H_1} F(yzh_1)dh_1 dz dy \\
&= \int_{G/H_2}\int_{H_2/H_1} f(yz)dz dy.
\end{aligned}
$$

□

**Exercise 7.(No $\mathrm{SL}_2(\mathbb{R})$-invariant Measure on $\mathrm{SL}_2(\mathbb{R})/P$):**

Let $G = \mathrm{SL}_2(\mathbb{R})$ and $P$ be the subgroup of upper triangular matrices. Show directly that there is no (non-trivial) finite $G$-invariant measure on $G/P$.

Hint: Identify $G/P \cong \mathbb{S}^1 \cong \mathbb{R} \cup \{\infty\}$ with the unit circle and consider a rotation

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

and a translation

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

**Solution:** Recall that $G = \mathrm{SL}(2,\mathbb{R})$ acts on the upper half plane $\mathbb{H} = \{z \in \mathbb{C} \,|\, \mathrm{Im}\, z > 0\} \subset \hat{\mathbb{C}}$ and its boundary $\partial\mathbb{H} = \mathbb{R} \cup \{\infty\} \subset \hat{\mathbb{C}}$ via Möbius transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.z = \frac{az+b}{cz+d}.$$

Note that $\mathrm{SL}(2,\mathbb{R})$ acts transitively on $\partial\mathbb{H}$ and the stabilizer of $\infty$ is the subgroup of upper triangular matrices $P$. We may therefore identify $G/P \cong \mathbb{R} \cup \{\infty\}$.

Suppose there is a finite $G$-invariant measure $m$ on $G/P \cong \mathbb{R} \cup \{\infty\}$. Consider the restriction $\mu = m|_{\mathbb{R}}$ of this measure to the real line. Observe that $G$ acts on the real line via translations

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.\xi = \xi + t, \qquad \xi, t \in \mathbb{R},$$

such that $\mu$ is in particular a translation invariant measure on $\mathbb{R}$, i.e. $\mu$ is a Haar measure on $\mathbb{R}$. By uniqueness of Haar measures $\mu$ must be a multiple of the Lebesgue measure on $\mathbb{R}$. Since $m$ is finite $\mu$ is the zero measure. That means that $m$ is a positive multiple of the dirac measure at $\infty$, i.e. $m = \lambda \cdot \delta_\infty$ for some $\lambda > 0$. Now consider the rotation

$$i(z) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.z = -\frac{1}{z}$$

that sends $\infty$ to 0. By $G$-invariance we must have

$$\lambda \cdot \delta_\infty = i_*(\lambda \cdot \delta_\infty) = \lambda \cdot \delta_0$$

such that $\lambda = 0$; in contradiction to our assumption. $\square$