Solution Exercise Sheet 4

Exercise 1.(Related Vector Fields):

Let M, N be smooth manifolds and let $\varphi : M \to N$ be a smooth map. Recall that two vector fields $X \in \mathrm{Vect}(M)$, $X' \in \mathrm{Vect}(N)$ are called φ-related if

$$d_p\varphi(X_p) = X'_{\varphi(p)}$$

for every $p \in M$.

Show that $[X,Y]$ is φ-related to $[X',Y']$ if $X \in \mathrm{Vect}(M)$ is φ-related to $X' \in \mathrm{Vect}(N)$ and $Y \in \mathrm{Vect}(M)$ is φ-related to $Y' \in \mathrm{Vect}(N)$.

Solution: Let $f \in C^\infty(N)$ be a smooth function on N and X, X', Y, Y' vector fields on M and N as above.

Let $p \in M$. We compute

$$\begin{aligned}
[X',Y']_{\varphi(p)}f &= X'_{\varphi(p)}(Y'f) - Y'_{\varphi(p)}(X'f) \\
&= (d_p\varphi(X_p))(Y'f) - (d_p\varphi(Y_p))(X'f) \\
&= X_p((Y'f)\circ\varphi) - Y_p((X'f)\circ\varphi).
\end{aligned}$$

Now, note that

$$(Y'f)(\varphi(q)) = Y'_{\varphi(q)}f = (d_q\varphi(Y_q))f = Y_q(f\circ\varphi), \qquad \forall q \in M,$$

because Y and Y' are φ-related and analogously

$$(X'f)(\varphi(q)) = X'_{\varphi(q)}f = (d_q\varphi(X_q))f = X_q(f\circ\varphi), \qquad \forall q \in M.$$

Therefore

$$\begin{aligned}
X_p((Y'f)\circ\varphi) - Y_p((X'f)\circ\varphi) &= X_p(Y(f\circ\varphi)) - Y_p(X(f\circ\varphi)) \\
&= [X,Y]_p(f\circ\varphi) \\
&= d_p\varphi([X,Y])f.
\end{aligned}$$

This shows that

$$d_p\varphi([X,Y]) = [X',Y']_{\varphi(p)},$$

i.e. $[X,Y]$ and $[X',Y']$ are φ-related.

□

Exercise 2.(Leibniz Rule):

Let $A,B : (-\varepsilon,\varepsilon) \to \mathbb{R}^{n\times n}$ be smooth curves and define $\varphi : (-\varepsilon,\varepsilon) \to \mathbb{R}^{n\times n}$ as the product $\varphi(t) := A(t)B(t)$. Show that

$$\varphi'(t) = A'(t)B(t) + A(t)B'(t)$$

for every $t \in (-\varepsilon,\varepsilon)$.

Solution: Note that the ij-entry of $\varphi(t)$ is

$$\varphi_{ij}(t) = \sum_{k=1}^{n} A_{ik}(t)B_{kj}(t)$$

for every $t \in (-\varepsilon,\varepsilon)$.

Differentiating each entry yields

$$\begin{aligned}\varphi'_{ij}(t) &= \sum_{k=1}^{n} A'_{ik}(t)B_{kj}(t) + \sum_{k=1}^{n} A_{ik}(t)B'_{kj}(t)\\ &= (A'(t)B(t))_{ij} + (A(t)B'(t))_{ij} \qquad \forall t \in (-\varepsilon,\varepsilon)\end{aligned}$$

such that

$$\varphi'(t) = A'(t)B(t) + A(t)B'(t)$$

as claimed. □

Exercise 3.(Some Lie Algebras):

a) Let M, N be smooth manifolds and let $f : M \to N$ be a smooth map of constant rank r. By the constant rank theorem we know that the level set $L = f^{-1}(q)$ is a regular submanifold of M of dimension $\dim M - r$ for every $q \in N$. Show that one may canonically identify

$$T_pL \cong \ker d_pf$$

for every $p \in L = f^{-1}(q)$.

Solution: Since $L = f^{-1}(q)$ is a regular submanifold of M we may think of the tangent space T_pL as a subspace of the tangent space T_pM. We will first show that $T_pL \subseteq \ker d_pf$. Let $v \in T_pL$ and let $\gamma : (-\varepsilon,\varepsilon) \to L = f^{-1}(q)$ be a smooth curve in L such that $\gamma(0) = p$ and $\gamma'(0) = v$. Then $f(\gamma(t)) = q$ for all $t \in (-\varepsilon,\varepsilon)$, i.e. $f \circ \gamma$ is the constant curve. It follows that

$$d_p f(v) = d_{\gamma(0)} f(\gamma'(0)) = \left.\frac{d}{dt}\right|_{t=0} f(\gamma(t)) = 0.$$

In particular, $v \in \ker d_p f$ as claimed.

Finally, note that $\ker d_p f$ is a subspace of T_pM of dimension

$$\dim \ker d_p f = \dim T_pM - \operatorname{rank} d_p f = \dim M - r = \dim L = \dim T_pL.$$

Therefore T_pL is a linear subspace of $\ker d_p f$ of maximal dimension such that $T_pL = \ker d_p f$. $\square$

b) Use part a) to compute the Lie algebras of the Lie groups $O(n,\mathbb{R})$, $O(p,q)$, $U(n)$, $Sp(2n,\mathbb{C})$, $B(n)$ and $N(n)$ where $B(n)$ is the group of real invertible upper triangular matrices and $N(n)$ is the subgroup of $B(n)$ with only ones on the diagonal.

Solution: Note that all of the listed Lie groups are subgroups of $\mathrm{GL}(n,\mathbb{K})$ that are also regular submanifolds ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). In particular the inclusion maps yield injective Lie algebra homomorphisms. This implies that the corresponding Lie algebras can be canonically identified with Lie subalgebras of $\mathfrak{gl}_n\mathbb{K}$. Hence the Lie bracket will be given by the ambient Lie bracket $[\cdot,\cdot]$ of $\mathfrak{gl}_n\mathbb{K}$. Identifying $\mathfrak{gl}_n\mathbb{K} \cong T_I\,\mathrm{GL}(n,\mathbb{K}) \cong \mathbb{K}^{n\times n}$ the Lie bracket is given by the commutator

$$[A,B] = AB - BA$$

as was proved in the lecture.

(i) $O(n,\mathbb{R})$: Consider the function $f_1 : \mathrm{GL}(n,\mathbb{R}) \to \mathbb{R}^{n\times n}$ given by

$$f_1(A) = A^T A$$

for every $A \in \mathrm{GL}(n,\mathbb{R})$. It is easy to check that f_1 has constant rank and that

$$O(n) = f_1^{-1}(I).$$

By part a)

$$\mathfrak{o}(n) := \mathrm{Lie}(O(n)) \cong T_I O(n) \cong \ker d_I f_1 < \mathfrak{gl}_n\mathbb{R}.$$

Let $X \in \mathbb{R}^{n\times n} \cong T_I\,\mathrm{GL}(n,\mathbb{R})$. We compute

$$\begin{aligned} d_I f_1(X) &= \left.\frac{d}{dt}\right|_{t=0} (I+tX)^t(I+tX) \\ &= X^t + X \end{aligned}$$

where we have used exercise 2 in the last equality. Therefore

$$\mathfrak{o}(n) = \{X \in \mathfrak{gl}_n\mathbb{R} : X^t + X = 0\}.$$

(ii) $O(p,q)$: Consider the function $f_2 : \mathrm{GL}(n,\mathbb{R}) \to \mathbb{R}^{n\times n}$ given by

$$f_2(A) = A^T I_{p,q} A$$

for every $A \in \mathrm{GL}(n,\mathbb{R})$, where

$$I_{p,q} = \mathrm{diag}(\underbrace{1,\ldots,1}_{p\text{-times}},\underbrace{-1,\ldots,-1}_{q\text{-times}}).$$

It is easy to check that f_2 has constant rank and that

$$O(p,q) = f_2^{-1}(I_{p,q}).$$

By part a)

$$\mathfrak{o}(p,q) := \mathrm{Lie}(O(p,q)) \cong T_I O(p,q) \cong \ker d_I f_2 < \mathfrak{gl}_n\mathbb{R}.$$

Let $X \in \mathbb{R}^{n\times n} \cong T_I\,\mathrm{GL}(n,\mathbb{R})$. We compute

$$\begin{aligned} d_I f_2(X) &= \frac{d}{dt}\bigg|_{t=0} (I+tX)^t I_{p,q}(I+tX) \\ &= X^t I_{p,q} + I_{p,q}X \end{aligned}$$

where we have used exercise 2 in the last equality. Therefore

$$\mathfrak{o}(p,q) = \{X \in \mathfrak{gl}_n\mathbb{R} : X^t I_{p,q} + I_{p,q}X = 0\}.$$

(iii) $U(n)$: Consider the function $f_3 : \mathrm{GL}(n,\mathbb{C}) \to \mathbb{C}^{n\times n}$ given by

$$f_3(A) = A^* A$$

for every $A \in \mathrm{GL}(n,\mathbb{C})$. It is easy to check that f_3 has constant rank and that

$$U(n) = f_3^{-1}(I).$$

By part a)

$$\mathfrak{u}(n) := \mathrm{Lie}(U(n)) \cong T_I U(n) \cong \ker d_I f_3 < \mathfrak{gl}_n\mathbb{C}.$$

Let $X \in \mathbb{C}^{n\times n} \cong T_I \operatorname{GL}(n,\mathbb{C})$. We compute

$$\begin{aligned} d_I f_3(X) &= \left.\frac{d}{dt}\right|_{t=0} (I+tX)^*(I+tX) \\ &= X^* + X \end{aligned}$$

where we have used exercise 2 in the last equality. Therefore

$$\mathfrak{u}(n) = \{X \in \mathfrak{gl}_n\mathbb{C} : X^* + X = 0\}.$$

(iv) $\operatorname{Sp}(2n,\mathbb{C})$: Consider the function $f_4 : \operatorname{GL}(n,\mathbb{C}) \to \mathbb{C}^{n\times n}$ given by

$$f_4(A) = A^t F A$$

for every $A \in \operatorname{GL}(n,\mathbb{C})$ where

$$F = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

It is easy to check that f_4 has constant rank and that

$$\operatorname{Sp}(2n,\mathbb{C}) = f_4^{-1}(F).$$

By part a)

$$\mathfrak{sp}(2n,\mathbb{C}) := \operatorname{Lie}(\operatorname{Sp}(2n,\mathbb{C})) \cong T_I \operatorname{Sp}(2n,\mathbb{C}) \cong \ker d_I f_4 < \mathfrak{gl}_n\mathbb{C}.$$

Let $X \in \mathbb{C}^{n\times n} \cong T_I \operatorname{GL}(n,\mathbb{C})$. We compute

$$\begin{aligned} d_I f_f(X) &= \left.\frac{d}{dt}\right|_{t=0} (I+tX)^t F(I+tX) \\ &= X^t F + FX \end{aligned}$$

where we have used exercise 2 in the last equality. Therefore

$$\mathfrak{sp}(2n,\mathbb{C}) = \{X \in \mathfrak{gl}_n\mathbb{C} : X^t F + FX = 0\}.$$

(v) $B(n)$: Consider the function $f_5 : \operatorname{GL}(n,\mathbb{R}) \to \mathbb{R}^{n\times n}$ given by

$$f_5(A) = \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ A_{21} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ A_{n1} & \cdots & A_{n,n-1} & 0 \end{pmatrix}$$

for every $A \in \operatorname{GL}(n,\mathbb{R})$. It is easy to check that f_5 has constant rank and that

$$B(n) = f_5^{-1}(0).$$

By part a)

$$\mathfrak{b}(n) := \mathrm{Lie}(B(n)) \cong T_I B(n) \cong \ker d_I f_5 < \mathfrak{gl}_n \mathbb{R}.$$

Let $X \in \mathbb{R}^{n\times n} \cong T_I\,\mathrm{GL}(n,\mathbb{R})$. We compute

$$\begin{aligned} d_I f_5(X) &= \frac{d}{dt}\bigg|_{t=0} f_5(I+tX) \\ &= \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ X_{21} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ X_{n1} & \cdots & X_{n,n-1} & 0 \end{pmatrix}. \end{aligned}$$

Therefore

$$\mathfrak{b}(n) = \left\{ \begin{pmatrix} X_{11} & \cdots & X_{1n} \\ & \ddots & \vdots \\ 0 & & X_{nn} \end{pmatrix} \in \mathbb{R}^{n\times n} \right\}.$$

(vi) $N(n)$: Consider the function $f_6 : \mathrm{GL}(n,\mathbb{R}) \to \mathbb{R}^{n\times n}$ given by

$$f_6(A) = \begin{pmatrix} X_{11} & & 0 \\ \vdots & \ddots & \\ X_{n1} & \cdots & X_{nn} \end{pmatrix}$$

for every $A \in \mathrm{GL}(n,\mathbb{R})$. It is easy to check that f_6 has constant rank and that

$$N(n) = f_6^{-1}(I).$$

By part a)

$$\mathfrak{n}(n) := \mathrm{Lie}(N(n)) \cong T_I N(n) \cong \ker d_I f_6 < \mathfrak{gl}_n \mathbb{R}.$$

Let $X \in \mathbb{R}^{n\times n} \cong T_I\,\mathrm{GL}(n,\mathbb{R})$. We compute

$$\begin{aligned} d_I f_6(X) &= \frac{d}{dt}\bigg|_{t=0} f_6(I+tX) \\ &= \begin{pmatrix} X_{11} & & 0 \\ \vdots & \ddots & \\ X_{n1} & \cdots & X_{nn} \end{pmatrix}. \end{aligned}$$

Therefore

$$\mathfrak{n}(n) = \left\{ \begin{pmatrix} 0 & * & * \\ \vdots & \ddots & * \\ 0 & \cdots & 0 \end{pmatrix} \in \mathbb{R}^{n\times n} \right\}.$$

□

Exercise 4.(Easy Direction of Frobenius' Theorem):

Let M be a smooth manifold and let $\mathcal{D}$ be a distribution on M. Show that $\mathcal{D}$ is involutive if it is completely integrable.

Solution: Let $U \subset M$ be an open set and $\{X_1,\dots,X_n\}$ a local basis of $\mathcal{D}$ defined on U. Further, let $q \in U$ and suppose q is contained in an integral submanifold $\varphi : N \hookrightarrow M$ of $\mathcal{D}$ such that $d_p\varphi(T_pN) = \mathcal{D}_p$ for every $p \in N$, where $\varphi : N \hookrightarrow M$ is an injective immersion. Let $p \in \varphi^{-1}(q)$ and choose open neighborhoods $V' \subset N$ about p and $U' \subset U$ about q such that $\varphi|_{V'} : V' \to U'$ is a smooth embedding. By using a local slice chart it is easy to see that the vector fields $\{Y_1,\dots,Y_n\}$ defined via

$$d_{p'}\varphi(Y_i) = (X_i)_{\varphi(p')} \qquad \forall p' \in V' \forall i = 1,\dots,n \tag{$\star\star$}$$

are smooth vector fields on $V' \subset N$. Here we have used that $\{(X_1)_{\varphi(p')},\dots,(X_n)_{\varphi(p')}\}$ is a basis of $\mathcal{D}_{\varphi(p')} = d_{p'}\varphi(T_{p'}N)$ and that the differential of $d_{p'}\varphi$ is injective for every $p' \in V'$. Note that ($\star\star$) means that Y_i is φ-related to X_i for every $i = 1,\dots,n$. By exercise 1 also $[Y_i,Y_j]$ is φ-related to $[X_i,X_j]$, i.e.

$$[X_i,X_j]_{\varphi(p')} = d_{p'}\varphi[Y_i,Y_j]_{p'},$$

for every $i,j = 1,\dots,n$. Because $\{Y_1,\dots,Y_n\}$ are smooth vector fields on $V' \subset N$ also $[Y_i,Y_j]_{p'}$ is a smooth vector field on $V' \subset N$. This implies that $[X_i,X_j]_{\varphi(p')} \in d_{p'}\varphi(T_{p'}N) = \mathcal{D}_{\varphi(p')}$ for every $p' \in V'$; in particular $[X_i,X_j]_q \in \mathcal{D}_q$. Therefore $\mathcal{D}$ is involutive. □

Exercise 5.(Distributions and Lie Subalgebras):

a) Let M be a smooth manifold, $X,Y \in \mathrm{Vect}(M)$ vector fields on M, and $f,g \in C^\infty(M)$ smooth functions. Show that

$$[fX,gY] = fg[X,Y] + f(Xg)Y - g(Yf)X.$$

Solution: Let $h \in C^\infty(M)$ and $p \in M$. We compute

$$\begin{aligned}([fX,gY]_ph) &= f(p)X_p(g(Yh)) - g(p)Y_p(f(Xh))\\ &= f(p)(X_pg)(Y_ph) + f(p)g(p)X_p(Yh)\\ &\quad - g(p)(Y_pf)(X_ph) - g(p)f(p)Y_p(Xh)\\ &= f(p)g(p)([X,Y]_ph) + f(p)(X_pg)(Y_ph) - g(p)(Y_pf)(X_ph).\end{aligned}$$

□

b) Show that the Lie algebra $\mathfrak{h}$ of a Lie subgroup H of a Lie group G determines a left-invariant involutive distribution.

<u>Remark:</u> Part a) is not necessarily needed for part b).

Solution: Let $\iota : H \hookrightarrow G$ be a Lie subgroup and let $X_1, \dots, X_n$ be a basis of $T_eH \cong \mathfrak{h}$. We define smooth left-invariant vector fields $Y_1, \dots, Y_n$ on G via

$$(Y_i)_g = d_eL_g(d_e\iota X_i)$$

for every $g \in G$, $i = 1, \dots, n$. These clearly define a global basis of the left-invariant distribution $\mathcal{D} = \operatorname{span}\{Y_1, \dots, Y_n\} \subset TG$ on G.

We need to see that $\mathcal{D}$ is involutive. Observe that Y_i is L_g-related to itself for every $g \in G$ by definition. By exercise 1 also $[Y_i, Y_j]$ is L_g-related to itself such that

$$[Y_i, Y_j]_g = [Y_i, Y_j]_{L_g(e)} = d_eL_g([Y_i, Y_j]_e)$$

for every $g \in G$. Further Y_i is ι-related to X_i by definition. Therefore also $[Y_i, Y_j]$ is ι-related to $[X_i, X_j]$ such that

$$[Y_i, Y_j]_e = [Y_i, Y_j]_{\iota(e)} = d_e\iota[X_i, X_j]_e \in \mathcal{D}_e.$$

Hence,

$$[Y_i, Y_j]_g = d_eL_g([Y_i, Y_j]_e) \in d_eL_g(\mathcal{D}_e) = \mathcal{D}_g$$

by left-invariance. This shows that $\mathcal{D}$ is involutive. □

Exercise 6.(Functions with values in immersed submanifolds):

Let M', M, N be smooth manifolds and let $\iota\colon N \hookrightarrow M$ be an injective immersion, i.e. ι is an injective smooth map whose differential is injective. Further, let $f : M' \to M$ be a smooth map with $f(M) \subseteq \iota(N)$.

Show that $\iota^{-1} \circ f : M' \to N$ is smooth if it is continuous.

Solution: Let $x \in M'$, let $y = f(x) \in M$ and let $z = \iota^{-1}(y) \in N$. Because ι is an immersion there are open neighborhoods $W \subseteq N, V \subseteq M$ about z, y resp. and charts $\xi\colon W \to \mathbb{R}^k$, $\psi\colon V \to \mathbb{R}^n$ such that $\iota(W) \subseteq V$ and

$$j(x_1, \dots, x_k) := (\psi \circ \iota \circ \xi^{-1})(x_1, \dots, x_k) = (x_1, \dots, x_k, 0, \dots, 0) \in \mathbb{R}^k \times \{0\} \subseteq \mathbb{R}^n$$

for all $(x_1, \dots, x_k) \in \mathbb{R}^k$, i.e. there is a slice chart for N.

Moreover, consider the open set $(\iota^{-1} \circ f)^{-1}(W) = f^{-1}(\iota(W)) \subseteq M'$ which contains an open neighborhood U of $x \in M'$ with a chart $\varphi\colon U \to \mathbb{R}^m$. Because $f(U) \subseteq \iota(W) \subseteq V$, we have

$$\begin{tikzcd}
W \arrow[r, hook, "\iota|_W"] \arrow[d, "\xi"] & V \arrow[d, "\psi"] & U \arrow[l, "f|_U"'] \arrow[d, "\varphi"] \\
\mathbb{R}^k \arrow[r, bend left, "j"] & \mathbb{R}^k \times \{0\} \arrow[l, bend left, "\pi"] & \mathbb{R}^m, \arrow[l]
\end{tikzcd}$$

where $\pi\colon \mathbb{R}^n \to \mathbb{R}^k$ is the projection on the first k-coordinates. This shows that $\iota^{-1} \circ f|_U$ is smooth in local charts about x and $z = \iota^{-1}(f(x))$. Because $x \in M'$ was arbitrary, this shows that $\iota^{-1} \circ f$ is smooth. $\square$