Solution Exercise Sheet 7

Exercise 1.(Ideals and Quotients of Nilpotent Lie Algebras):

Let $\mathfrak{g}$ be a Lie algebra and let $\mathfrak{h} \trianglelefteq \mathfrak{g}$ be an ideal. We have already seen that, $\mathfrak{g}$ is solvable if and only if $\mathfrak{h}$ and $\mathfrak{g}/\mathfrak{h}$ are solvable.

Show that such a statement cannot hold for nilpotent $\mathfrak{g}$!

Solution: The analogous statement for nilpotent Lie algebras cannot be true, because if it were we could show that any solvable Lie algebra is nilpotent by induction on $\dim \mathfrak{g}$. In fact, let us assume that $\mathfrak{g}$ is solvable. If $\mathfrak{g}$ is one-dimensional then it is certainly nilpotent. If $\dim \mathfrak{g} > 1$, given any ideal $\mathfrak{h} \subset \mathfrak{g}$, both $\mathfrak{h}$ and $\mathfrak{g}/\mathfrak{h}$ are solvable. Since their dimension is smaller than the dimension of $\mathfrak{g}$, they would be nilpotent by inductive hypothesis and hence $\mathfrak{g}$ would be nilpotent. □

Exercise 2.(Adjoint of nilpotent elements):

Let $\mathfrak{g} \leq \mathfrak{gl}_n(\mathbb{C})$ be a Lie subalgebra.

Show that, if $X \in \mathfrak{g}$ is nilpotent then $\mathrm{ad}(X) \in \mathfrak{gl}(\mathfrak{g})$ is nilpotent.

Solution: This will follow from the following formula

$$\mathrm{ad}(X)^n(Y) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} X^{n-k} Y X^k \tag{1}$$

for every $X, Y \in \mathfrak{g}$, $n \geq 0$.

Indeed, $X \in \mathfrak{g}$ is nilpotent if and only if $X^m = 0$ for some $m \in \mathbb{N}$. Then, by the above formula (1), $\mathrm{ad}(X)^{2m}(Y) = 0$ for every $Y \in \mathfrak{g}$.

We will prove (1) by induction on n. For $n = 0$ there is nothing to show. So, let us assume that (1) holds for n and we want to prove it for $n + 1$. This is a direct computation:

$$
\begin{aligned}
\operatorname{ad}(X)^{n+1}(Y) &= \operatorname{ad}(X)(\operatorname{ad}(X)^n(Y)) \\
&= X \cdot \operatorname{ad}(X)^n(Y) - \operatorname{ad}(X)^n(Y) \cdot X \\
&= X \cdot \left(\sum_{k=0}^{n} (-1)^k \binom{n}{k} X^{n-k} Y X^k \right) - \left(\sum_{k=0}^{n} (-1)^k \binom{n}{k} X^{n-k} Y X^k \right) \cdot X \\
&= X^{n+1} Y + \sum_{k=1}^{n} (-1)^k \binom{n}{k} X^{n-k+1} Y X^k - \sum_{k=0}^{n-1} (-1)^k \binom{n}{k} X^{n-k} Y X^{k+1} + (-1)^{n+1} Y X^{n+1} \\
&= X^{n+1} Y + \sum_{k=1}^{n} (-1)^k \left(\binom{n}{k} + \binom{n}{k-1} \right) X^{n-k+1} Y X^k + (-1)^{n+1} Y X^{n+1} \\
&= X^{n+1} Y + \sum_{k=1}^{n} (-1)^k \binom{n+1}{k} X^{n-k+1} Y X^k + (-1)^{n+1} Y X^{n+1} \\
&= \sum_{k=0}^{n+1} (-1)^k \binom{n}{k} X^{n+1-k} Y X^k
\end{aligned}
$$

□

Exercise 3.(Cartan's criterion for solvability):

Let $\mathfrak{g}$ be a Lie algebra with Killing form $B_{\mathfrak{g}}$.

Show that $\mathfrak{g}$ is solvable if and only if $B_{\mathfrak{g}}|_{\mathfrak{g}^{(1)} \times \mathfrak{g}^{(1)}} = 0$.

Hint: One direction is an easy verification. For the other direction you can use a theorem from class in conjunction with the fact that a Lie algebra $\mathfrak{g}$ with an ideal $\mathfrak{h} \trianglelefteq \mathfrak{g}$ is solvable if and only if both $\mathfrak{h}$ and $\mathfrak{g}/\mathfrak{h}$ are solvable.

Solution: ($\Rightarrow$) Suppose that $\mathfrak{g}$ is solvable. Then $\mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}]$ is nilpotent. Hence, $\operatorname{ad}(\mathfrak{g}^{(1)})$ is strictly upper triangular. This implies that $B_{\mathfrak{g}}|_{\mathfrak{g}^{(1)} \times \mathfrak{g}^{(1)}} = B_{\mathfrak{g}^{(1)}} = 0$.

($\Leftarrow$) From a theorem in class we obtain $\operatorname{ad}(\mathfrak{g}))^{(1)} = \operatorname{ad}(\mathfrak{g}^{(1)})$, if $X, Y \in \mathfrak{g}^{(1)}$ and $0 = B_{\mathfrak{g}}(X, Y) = \operatorname{tr}(\operatorname{ad}(X)\operatorname{ad}(Y))$, then $[\operatorname{ad}(\mathfrak{g}^{(1)}), \operatorname{ad}(\mathfrak{g}^{(1)})] = \operatorname{ad}([\mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}]) = \operatorname{ad}(\mathfrak{g}^{(2)})$ is strictly upper triangular and hence nilpotent, hence solvable. We need to go show that $\mathfrak{g}$ is solvable.

Since $\operatorname{ad}(\mathfrak{g}^{(2)})$ is a solvable ideal in $\operatorname{ad}(\mathfrak{g}^{(1)})$ and $\operatorname{ad}(\mathfrak{g}^{(1)})/\operatorname{ad}(\mathfrak{g}^{(2)}) = \operatorname{ad}(\mathfrak{g}^{(1)}/\mathfrak{g}^{(2)})$ is Abelian, hence solvable, $\operatorname{ad}(\mathfrak{g}^{(1)})$ is solvable. Analogously, $\operatorname{ad}(\mathfrak{g}^{(1)})$ is a solvable ideal in $\operatorname{ad}(\mathfrak{g})$ and $\operatorname{ad}(\mathfrak{g})/\operatorname{ad}(\mathfrak{g}^{(1)}) = \operatorname{ad}(\mathfrak{g}/\mathfrak{g}^{(1)})$ is Abelian, hence solvable, so $\operatorname{ad}(\mathfrak{g})$ is solvable. Finally the short exact sequence $0 \to Z(\mathfrak{g}) \to \mathfrak{g} \to \operatorname{ad}(\mathfrak{g}) \to 0$ shows that $\mathfrak{g}$ is solvable. □

Exercise 4.(Direct sums of simple ideals):

Let $\mathfrak{g} = \bigoplus_{i \in I} \mathfrak{g}_i$ be the direct sum of simple ideals. Then any ideal $\mathfrak{h} \trianglelefteq \mathfrak{g}$ is of the

form $\mathfrak{h} = \bigoplus_{j\in J} \mathfrak{g}_j$ with $J \subset I$.
Remark: This implies immediately:

(i) Any semisimple Lie algebras has a finite number of ideals.

(ii) Any connected semisimple Lie group with finite center has a finite number of connected normal subgroups.

Solution: Let $J \subset I$ be the smallest subset such that $\mathfrak{h} \subseteq \bigoplus_{i\in J} \mathfrak{g}_i$. We are going to show that there is equality. Let $i \in J$. Then $[\mathfrak{h}, \mathfrak{g}_i] \subseteq \mathfrak{g}_i$, since $\mathfrak{g}_i$ is an ideal; moreover, since $[\mathfrak{h}, \mathfrak{g}_i]$ is an ideal, either $[\mathfrak{h}, \mathfrak{g}_i] = \mathfrak{g}_i$ or $[\mathfrak{h}, \mathfrak{g}_i] = \{0\}$. We will show that $[\mathfrak{h}, \mathfrak{g}_i] \neq \{0\}$ for every $i \in J$, so that $\mathfrak{g}_i = [\mathfrak{h}, \mathfrak{g}_i] \subset \mathfrak{h}$, which implies that $\mathfrak{h} = \bigoplus_{i\in J} \mathfrak{g}_i$.

To see that $[\mathfrak{h}, \mathfrak{g}_i] \neq \{0\}$, let us suppose by contradiction that $[\mathfrak{h}, \mathfrak{g}_i] = \{0\}$. Then if $X \in \mathfrak{h}$ we can write $X = X_1 + \cdots + X_n$ with $|J| = n$. In particular from $[X, \mathfrak{g}_i] = \{0\}$ it follows that $[X_i, \mathfrak{g}_i] = \{0\}$. Thus $\mathfrak{h} \cap \mathfrak{g}_i \subset Z_{\mathfrak{g}_i}(\mathfrak{g}_i) = 0$, which contradicts the minimality of J. □

Exercise 5.(Characterization of Semi-Simplicity):

Let $\mathfrak{g}$ be a Lie algebra. Show that the following statements are equivalent:

(i) $\mathfrak{g}$ is semisimple;

(ii) $\mathfrak{g}$ has no non-trivial abelian ideals;

(iii) $\mathfrak{g}$ has no non-trivial solvable ideals.

Solution: We will first show that b) is equivalent to c). Then we will see that a) is equivalent to c).

b) $\Longleftrightarrow$ c):

If $\mathfrak{g}$ has a non-trivial abelian ideal then this ideal is clearly a non-trivial solvable ideal. Hence c) implies b) indirectly.

On the other hand, if $\mathfrak{h}$ is a non-trivial solvable ideal in $\mathfrak{g}$ with solvability length n then $\mathfrak{h}^{(n-1)} \neq 0$ is a non-trivial ideal in $\mathfrak{g}$ that is also abelian because

$$\mathfrak{h}^{(n-1)}/\mathfrak{h}^{(n)} = \mathfrak{h}^{(n-1)}$$

is abelian. Therefore, b) implies c) indirectly.

a) $\Longleftrightarrow$ c):

If $\mathfrak{g}$ has a non-trivial solvable ideal it has also a non-trivial abelian ideal $\mathfrak{a}$ as we have just seen. Because $\mathfrak{a}$ is abelian $\mathrm{ad}_{\mathfrak{g}}(X)$ is of the form

$$\mathrm{ad}_{\mathfrak{g}}(X) = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$$

for every $X \in \mathfrak{a}$ where the first column and row correspond to a basis of $\mathfrak{a}$. Because $\mathfrak{a}$ is an ideal $\mathrm{ad}_{\mathfrak{g}}(Y)$ is of the form

$$\mathrm{ad}_{\mathfrak{g}}(Y) = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$

for every $Y \in \mathfrak{g}$. Hence

$$B_{\mathfrak{g}}(X,Y) = \mathrm{tr}(\mathrm{ad}_{\mathfrak{g}}(X)\,\mathrm{ad}_{\mathfrak{g}}(Y)) = \mathrm{tr}\left(\begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\right) = \mathrm{tr}\begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix} = 0$$

for every $X \in \mathfrak{a}, Y \in \mathfrak{g}$, and $B_{\mathfrak{g}}$ is degenerate. By Dieudonné $\mathfrak{g}$ is therefore not semisimple. This shows that a) implies c) indirectly.

Finally, suppose that $\mathfrak{g}$ is not semisimple. Therefore $\mathfrak{h} = \mathfrak{g}^{\perp} \neq \{0\}$ is a non-trivial ideal in $\mathfrak{g}$ by 7a). Let $X, Y \in \mathfrak{h}, Z \in \mathfrak{h}^{(1)}$. Then

$$B_{\mathfrak{g}}([X,Y],Z) = B_{\mathfrak{g}}(X,[Y,Z]) = 0.$$

This shows that $B_{\mathfrak{g}}$ vanishes when restricted to $\mathfrak{h}^{(1)} \times \mathfrak{h}^{(1)}$. By Cartan's criterion this implies that $\mathfrak{h}$ is a non-trivial solvable ideal. □

Exercise 6.(Semi-Simple Lie Algebras equal their commutator):

Let $\mathfrak{g}$ be a Lie algebra. Show that if $\mathfrak{g}$ is semisimple then $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$.

Solution: By Dieudonné the Killing form $B_{\mathfrak{g}}$ is non-degenerate. It is therefore sufficient to show that $\left(\mathfrak{g}^{(1)}\right)^{\perp} = \{0\}$. Let $Z \in \left(\mathfrak{g}^{(1)}\right)^{\perp}$. Then

$$0 = B_{\mathfrak{g}}(Z,[X,Y]) = B_{\mathfrak{g}}([Z,X],Y)$$

for all $X, Y \in \mathfrak{g}$. Because the Killing form is non-degenerate this implies that $[Z,X] = 0$ for all $X \in \mathfrak{g}$, or equivalently the map $\mathrm{ad}_{\mathfrak{g}}(Z)$ is zero everywhere. Thus

$$B_{\mathfrak{g}}(Y,Z) = \mathrm{tr}(\mathrm{ad}_{\mathfrak{g}}(Y)\,\mathrm{ad}_{\mathfrak{g}}(Z)) = 0$$

for every $Y \in \mathfrak{g}$. Because the Killing form is non-degenerate this implies that $Z = 0$. □