Introduction to Lie groups

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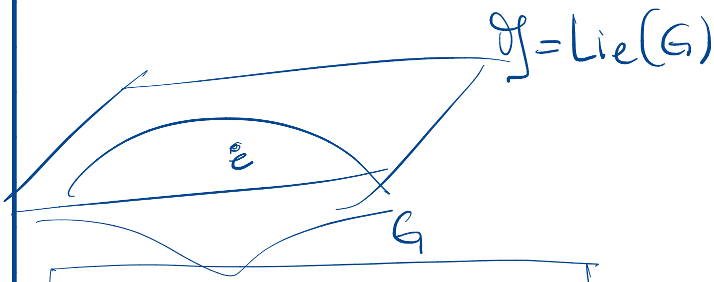
In week $2k-1$, get Pb set k to be handed in in week $2k+1$.

You may ask questions in the Ex. session in week $2k$ (otherwise devoted to discussion of Pb. set $k-1$)

Sophus Lie: continuous transf. gps.

Alg. eq. ⟷ Diff. eq.
Galois ⟷ Lie (?)



$\boxed{\text{Algebraic approach}}$

Geom. approach

Naive approach

Can consider only Lie gps $G < GL(n,\mathbb{R})$.

Lie gps as topological gps

Today: Top. gps & examples.

Tomorrow: Which of these ex. are (loc.) cpt.

Why loc. cpt. gps have the Haar measure!

1. Topological gps
2. Lie groups & their Lie algebras
3. Structure theory: essential results about nilpotent, solvable & semisimple Lie groups.
4. (?) Hands-on approach to algebraic gps.

Defn. A topological group G is a group endowed with a Hausdorff topology with respect to which the group operations

$$m: G\times G \to G,\ (g,h)\mapsto gh, \qquad i: G\to G,\ g\mapsto g^{-1}$$

are continuous.

Recall A top. space is Hausdorff if any two distinct pts have disjoint neighborhoods.

Rk The Zariski topology is not Hausdorff.

Ex. Any gp. with the discrete top. is a top. gp.

Ex $(\mathbb{R}^n,+)$ is a comm. top. gp. w.r.t. the Euclidean topology

Ex $(\mathbb{R}^*,\cdot)$, $(\mathbb{C}^*,\cdot)$ are Ab. top. gps. with respect to the top. induced by the Euclidean top.

Ex. $\mathbb{R}^{n^2}$ = $n\times n$ matrices with real coeff.

$GL(n,\mathbb{R}) := \{A\in\mathbb{R}^{n^2} : \det A\neq 0\}$ is open set in $\mathbb{R}^{n^2}$ $\rightsquigarrow$ it inherits the Euclidean topology.

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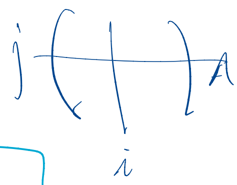
(*) $A,B\in GL(n,\mathbb{R}) \Rightarrow$

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj}$$

(**) $A\in GL(n,\mathbb{R}) \Rightarrow$

$$\Rightarrow (A^{-1})_{ij} = \frac{\det M_{ij}}{\det A}$$

M_{ij} = (i,j)-minor



$$\boxed{\begin{array}{l} \bullet\ (A_k)\in GL(n,\mathbb{R}) \\ A_k \to A \iff (A_k)_{ij}\to A_{ij} \end{array}}$$

$\Rightarrow$ $GL(n,\mathbb{R})$ is a top. gp with the Euclidean topology.

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Ex. Can replace $\mathbb{R}$ with any top. field k (i.e. add. mult. & inv. are cont.) e.g. $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}_p$, finite fields.

$GL(n,k)$ is a top. gp.

Ex. X cpt. Hausdorff space

$$\mathrm{Homeo}(X) := \{f: X\to X \text{ homeo}\}$$

is a top. gp. with the cpt.-open topology.

Recall (1) If X,Y are top. space, the compact-open top. on $C(X,Y)=\{f: X\to Y \text{ f cont.}\}$

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is the topology generated by the subbasis

$$S(C,U) := \{f\in C(X,Y) : f(C)\subset U\},$$

$C\subset X$ cpt., $U\subset Y$ open.

(2) If (Y,d) is a metric space $\Rightarrow$

$$B_C(f,\varepsilon) := \{g\in C(X,Y) : \sup_{x\in C} d(f(x),g(x))<\varepsilon\}$$

where C is cpt., $\varepsilon>0$, is a basis for the cpt-open top. on $C(X,Y)$

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Easy to see: If $(f_n) \subset C(X,Y)$ then $f_n \to f$ in the compact-open top. $\Leftrightarrow$ $f_n \to f$ in the top. of unf. conv. on cpt sets.

$\begin{pmatrix}\text{ptwise} \\ \text{convergence}\end{pmatrix} \Leftarrow \begin{pmatrix}\text{unif. conv.} \\ \text{on cpt. sets}\end{pmatrix} \Leftarrow \begin{pmatrix}\text{unf.} \\ \text{conv.}\end{pmatrix}$

$\Rightarrow$ discrete top. $\Rightarrow$ compact

Warning: Homeo(X) can be humongous if X is not cpt.

X loc. cpt. $\not\Rightarrow$ Homeo(X) top. gp.

X loc. cpt + loc. conn. $\Rightarrow$
$\Rightarrow$ Homeo(X) top. gp.

X proper metric space (closed balls of finite radius are cpt) $\Rightarrow$
$\Rightarrow$ Homeo(X) top. gp.

Ex X compact metric space.

$\mathrm{Iso}(X) := \{f \in \mathrm{Homeo}(X) : d(f(x), f(y)) = d(x,y) \ \forall x,y \in X\}$.

Then $\mathrm{Iso}(X) < \mathrm{Homeo}(X)$ is a closed sbgp $\Rightarrow$
$\Rightarrow$ top. gp.

Ex Before: top. gp. with the Eucl. an top.

But $GL(n,\mathbb{R}) < \mathrm{Homeo}(\mathbb{R}^n)$ ← metric space

$\Rightarrow$ the cpt. open top on Homeo$(\mathbb{R}^n)$ is the top of unf. conv. on cpt. sets.

If $(A_k) \subset GL(n,\mathbb{R})$, $A_k \to A$ unf. on cpt. sets $\Rightarrow$
A linear $\Rightarrow$ $GL(n,\mathbb{R})$ is a top. gp with resp. to the compact open top.

Ex. The cpt-open & the Euclidean top. on $GL(n,\mathbb{R})$

coincide.

Ex M diff. manifold,

$\mathrm{Diff}^r(M) := \{f : M \to M,\ f, f^{-1} \in C^r(M)\}$

not a top. gp. in the compact-open topology.
However the subspace of (f_n) : $\left.\begin{matrix} f_n \to f \\ f_n^{(k)} \to f^{(k)} \end{matrix}\right\}$ unf. on cpt. sets

$0 \le k \le r$ is a top. gp.

Ex Λ = partially ordered set, $(G_\lambda)_{\lambda \in \Lambda}$.
Let us assume that $\forall$

λ_1, λ_2 with $\lambda_1 \le \lambda_2$ $\exists$ homo

$$G_{\lambda_2} \xrightarrow{p_{\lambda_2,\lambda_1}} G_{\lambda_1}$$

$((G_\lambda)_{\lambda\in\Lambda}, p_{\lambda_2,\lambda_1})$ a <u>projective system</u>

Then the <u>inverse limit</u> of a proj. system is the unique smallest top. gp. G s.t. $\forall \lambda \in \Lambda$ $\exists$ homo $p_\lambda : G \to G_\lambda$ such that the diagram

$$\begin{tikzcd} G \arrow[r, "p_{\lambda_2}"] \arrow[dr, "p_{\lambda_1}"'] \arrow[dr, phantom, shift left=4, "\circlearrowleft"] & G_{\lambda_2} \arrow[d, "p_{\lambda_2,\lambda_1}"] \\ & G_{\lambda_1} \end{tikzcd}$$

$$p_{\lambda_2,\lambda_1} \circ p_{\lambda_2} = p_{\lambda_1}$$

Also

$$\varprojlim G_\lambda := \left\{ (x_\lambda)_{\lambda\in\Lambda} \in \prod_{\lambda\in\Lambda} G_\lambda : p_{\lambda_2,\lambda_1}(x_{\lambda_2}) = x_{\lambda_1} \right\}$$

(compatible sequences).

<u>Fact</u>: $G = \varprojlim G_\lambda$

If the (G_λ) are top. gps

$\Rightarrow \prod_{\lambda\in\Lambda} G_\lambda$ is a top. gp

$\Rightarrow \varprojlim G_\lambda$ top. gp.

<u>Facts</u> (1) (G_λ) cpt $\Rightarrow$

$\Rightarrow \varprojlim G_\lambda$ cpt

(2) (G_λ) discrete $\Rightarrow$

$\Rightarrow \varprojlim G_\lambda$ is totally disconnected

<u>Defn.</u> A <u>profinite</u> gp is the proj. limit of a proj. system consisting of finite gps.

Profinite $\Rightarrow$ compact & totally disconnected

For example

$$\left((\mathbb{Z}/p^n\mathbb{Z})_{n\in\mathbb{N}},\ p_{n,m} : \mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}/p^m\mathbb{Z} \right)$$

reduction mod p^m

The projective limit is the p-adic integers $\mathbb{Z}_p$; this is a compactification of $\mathbb{Z}$ (exercise)

<u>Ex</u> Important subgps of $GL(n,\mathbb{R})$

(1) $A = \left\{ \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} : \lambda_i \neq 0 \right\} \leq GL(n, \mathbb{R})$

$\cong \mathbb{R}^n$

Romeo + Romo

(2) $N = \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \right\} < GL(n, \mathbb{R})$

$\cong \mathbb{R}^{\frac{n(n-1)}{2}}$

Romeo

Not as gps since N is abelian only if $n \leq 2$

(3) $K = O(\mathbb{R}^n, \langle \cdot, \cdot \rangle) =$

$= \{ X \in GL(n, \mathbb{R}) : \langle Xv, Xw \rangle = \langle v, w \rangle \ \forall v, w \in \mathbb{R}^n \}$

$= \{ X \in GL(n, \mathbb{R}) : \|Xv\| = \|v\|, v \in \mathbb{R}^n \}$

$= \{ X \in GL(n, \mathbb{R}) : {}^tXX = \mathrm{Id}_n \}.$