Examples from yesterday:

$$A = \begin{pmatrix} \ddots & * \\ 0 & \ddots \end{pmatrix}, \quad N = \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix},$$

$K = O(n,\mathbb{R})$ = gp that preserves the inner product on $\mathbb{R}^n$.

Ex. V real vector space

$B: V \times V \to \mathbb{R}$

non-degenerate symm. bilinear fn

1) $\forall x \in V \ \exists y = y(x) \in V$ s.t.

$$B(x,y) \neq 0$$

2) $B(x,y) = B(y,x)$

3) $B(\alpha x, y) = \alpha B(x,y) = B(x, \alpha y)$

$\forall x, y \in V, \ \alpha \in \mathbb{R}$.

$Q(x) := B(x,x)$

$O(V,B) = \{A \in GL(V) : B(Ax,Ay) = B(x,y) \ \forall x,y \in V\}$.

orthogonal gp of (V,B) is a topological gp.

V real $\Rightarrow \exists p$, $0 \leq p \leq n$ s.t. we can find a basis of V w.r.t. B becomes

$$B_p(v,w) = -\sum_{j=1}^{p} v_j w_j + \sum_{j=p+1}^{n} v_j w_j$$

B is positive definite iff $p=0$

$$Q_p(v) = -\sum_{j=1}^{p} v_j^2 + \sum_{j=p+1}^{n} v_j^2$$

Sometimes we write

$$O(p,q) := O(V, B_p)$$

Rk: We take V to be f.d. In fact all our linear groups are f.d.

If V is complex then $(e_1, \dots, e_p, e_{p+1}, \dots, e_n) \rightsquigarrow$

$\rightsquigarrow (ie_1, \dots, ie_p, e_{p+1}, \dots, e_n)$

Then B_p becomes w.r.t. this new basis

$$B(v,w) = \sum_{j=1}^{n} v_j w_j$$

$\Rightarrow$ over $\mathbb{C}$ $\exists!$ orthog. gp. of a non-deg. symm. bil. form on a cplx, n-dim. v.sp.

$$O(n,\mathbb{C}) = O(V,B).$$

Ex: V complex vector space, $h: V \times V \to \mathbb{C}$ hermitian inner product, i.e. positive definite antisymmetric complex valued form that is linear w.r.t. the first variable and antilinear w.r.t. the second.

$U(V,h)$ = unitary group of (V,h)

$= \{X \in GL(V) : h(Xv, Xw) = h(v,w)\}$

$= \{X \in GL(V) : X^* = X^{-1}\}$.

where X^* is the adj. w.r.t. h.

If $X \in U(V,h) \Rightarrow |\det X| = 1$.

An example of such an h is

$$h : \mathbb{C}^n \times \mathbb{C}^n \longrightarrow \mathbb{C}$$

$$h(x,y) := \sum_{j=1}^{n} x_j \bar{y}_j .$$

and $U(\mathbb{C}^n, h) =: U(n)$.

Ex. Special linear gp

$$SL(n,k) = \{A \in GL(n,k) : \det A = 1\}$$

[k top. field
$k = \mathbb{R}, \mathbb{C}, \mathbb{Q}_p$, finite field]

$SO(n,\mathbb{R}) := O(n,\mathbb{R}) \cap SL(n,\mathbb{R})$
special orthog. gp.

$SO(p,q) := O(p,q) \cap SL(p+q,\mathbb{R})$

$SO(n,\mathbb{C}) := O(n,\mathbb{C}) \cap SL(n,\mathbb{C})$

$SU(n) := U(n) \cap SL(n,\mathbb{C})$

Ex: E normed vector space (not nec. f.d.), let $Iso(E)$ be the space of bijective cont. maps $T : E \to E$ that preserves the norm. Then $Iso(E)$ is a top. gp.

Ex $\mathcal{H}$ separable Hilbert space,

$\mathcal{U}(\mathcal{H})$ = space of continuous unitary operators on $\mathcal{H}$
$= \{U : \mathcal{H} \to \mathcal{H} : U^* = U^{-1}\}$.

Then $\mathcal{U}(\mathcal{H})$ is a topological group w.r.t. the strong operator topology.

Recall If E, F are normed spaces

$$B(E,F) := \{T : E \to F : T \text{ linear \& continuous}\}$$

with $\|T\| = \sup_{\|x\|=1} \|Tx\|$

Topologies on $B(E,F)$:

(1) $T_n \to T$ in the norm topology iff $\|T_n - T\| \to 0$

(2) $T_n \to T$ in the strong operator topology if
$\|T_n x - Tx\| \to 0 \quad \forall x \in E$

(3) $T_n \to T$ in the weak operator topology if
$\lambda(T_n x) \to \lambda(Tx) \quad \forall x \in E$
and $\forall \lambda \in F^*$.

- If E is a normed space over $k = \mathbb{R}, \mathbb{C}$ and $F = k$
$\Rightarrow B(E,k) = E^*$ and the weak-oper. top is the weak-* top. on E^*.
- If $E = F$ is f.d. $\dim E = n$
$\Rightarrow B(E,F) = B(E) \overset{\text{not nec. invertible}}{\supseteq}$
$\supseteq GL(E) \overset{\text{invertible}}{=} GL(n,k)$
- If $E = F = \mathcal{H}$ Hilbert space, then $Iso(E) = \mathcal{U}(\mathcal{H})$ and on $\mathcal{U}(\mathcal{H})$ the strong oper.

topology and the weak operator topology coincide.

Compactness & local compactness

Discrete grps

$(\mathbb{R}^N, +)$

$(\mathbb{R}^*, \cdot), (\mathbb{C}^*, \cdot)$

$GL(n,\mathbb{R}), GL(n,\mathbb{C})$ l.c. since open in $\mathbb{R}^{n^2}$, $\mathbb{C}^{n^2}$.

Ex. X cpt $\Rightarrow$ Homeo(X) top. gp. but not necessarily loc. cpt.

Exercise Homeo(S^1) is not locally compact and in fact Homeo(M) not l.c. for a cpt mfd.

Ex X metric space, Iso(X) is "as good" as X

X cpt $\Rightarrow$ Iso(X) cpt.

X l.c. $\overset{(*)}{\Rightarrow}$ Iso(X) l.c.

$\underbrace{\text{Iso}(X)}_{\text{equicont. family in } C(X,X)} \subset \text{Homeo}(X) \subset C(X,X)$

Arzelà - Ascoli thm $\Rightarrow$ Iso(X) (rel.) cpt.

Ex. $(*)$

Ex $O(p,q)$ is a top. grp as a subgrp of $GL(p+q,\mathbb{R})$.

To show:

$O(p,q)$ compact iff $p=0$

Pf $p=0 \Rightarrow O(0,n) = O(n,\mathbb{R})$.

$A \in O(n,\mathbb{R})$, $A = ((c_1), \dots, (c_n))$, where $c_j = Ae_j$, $1 \le j \le n$.

By defn ${}^tAA = Id_n \Rightarrow$

$\Rightarrow (\det A)^2 = 1 \Rightarrow \det A = \pm 1$

$\Rightarrow \|c_j\|^2 = 1$, $1 \le j \le n \Rightarrow$

$\Rightarrow |A_{ij}| \le 1 \Rightarrow O(n,\mathbb{R})$ is bounded in $\mathbb{R}^{n^2}$.

Since $O(n,\mathbb{R}) = \{A \in \mathbb{R}^{n^2} :$

$\langle Av, Aw\rangle = \langle v, w\rangle \ \forall v,w \in \mathbb{R}^n\}$

$\Rightarrow O(n,\mathbb{R})$ is closed in $\mathbb{R}^{n^2} \Rightarrow O(n,\mathbb{R})$ compact.

let us assume now that $pq \neq 0$

We write $O(p,q) =$

$= \{A \in \mathbb{R}^{n^2} : Q_p(Av) = Q_p(v) \ \forall v \in \mathbb{R}^n\}$

$$Q_p(v) = -\sum_{j=1}^{p} v_j^2 + \sum_{j=p+1}^{n} v_j^2$$

Look at $p=1$:

$$Q_1(v) = -v_1^2 + \sum_{j=2}^{n} v_j^2$$

w.r.t. $(e_1, \dots, e_n)$; change basis

$e_1' := e_2 - e_1$

$e_2' := e_2 + e_1$

$e_j' := e_j \qquad j = 3, \dots, n$

and denote V the v.s.

w.r.t. the new basis, on V the quadr. form Q_1 will become

$$Q_1'(v) = (v_2'-v_1')(v_2'+v_1') + \sum_{j=3}^{n} (v_j')^2$$

Then $A_t = \begin{pmatrix} t & 0 & 0 \\ 0 & 1/t & 0 \\ 0 & 0 & I_{n-2} \end{pmatrix}$ $\forall t \in \mathbb{R}$

satisfies $Q_1'(Av) = Q_1'(v)$

$\Rightarrow A \in O(V, Q_1')$

$\Rightarrow O(V, Q_1')$ is not compact since for $t \to \infty$, A_t leaves all compact sets.

Ex. This shows also that $SL(n,\mathbb{R})$ is not compact since $A_t \in SL(n,\mathbb{R})$.

Ex. Profinite gps are compact since inverse limits of a proj. system of cpt. gps.

Ex $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$

$\cong SO(2,\mathbb{R}) = O(2,\mathbb{R}) \cap SL(2,\mathbb{R})$

$$SO(2,\mathbb{R}) \longrightarrow \mathbb{T}$$
$$X \longmapsto X e_1$$

$$SO(2,\mathbb{R}) = \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} : \theta \in [0, 2\pi) \right\}$$

Warning: $O(n,\mathbb{C}) \neq U(n)$!

orth. gp. of bilinear sym. quadr. form

not compact

orth. gp. of an antisym. linear in first & antilinear in second pos. defn.

compact

$O(n,\mathbb{R})$

cpt.

p=1 $e_1' := e_2 - e_1$
$e_2' := e_2 + e_1$
$e_j' := e_j$ $\quad 3 \le j \le n$

p=2 $e_1' := e_2 - e_1$
$e_2' := e_2 + e_1$
$e_3' := e_4 - e_3$
$e_4' := e_4 + e_3$
$e_j' := e_j$ $\quad 5 \le j \le n$

Ex. $\mathcal{H}$ separable Hilbert space, $\mathcal{U}(\mathcal{H})$ is a top. gp.

Lemma $\mathcal{H}$ separable. $\mathcal{U}(\mathcal{H})$ is loc. cpt iff $\dim \mathcal{H}$ is finite, in which case $\mathcal{U}(\mathcal{H})$ is cpt.

Rk $\mathcal{U}(\mathcal{H})$ is never locally compact and not compact.

Pf ($\Leftarrow$) $\dim \mathcal{H} = n < \infty$.

$\mathcal{U}(\mathcal{H}) = U(n)$ cpt.

($\Rightarrow$) Let us consider a nbhd of $I \in \mathcal{U}(\mathcal{H})$.

$U_{F,\varepsilon} := \{ T \in \mathcal{U}(\mathcal{H}) : \|Tu - u\| < \varepsilon \ \ \forall u \in F \}$

where $\varepsilon > 0$ and $F \subset \mathcal{H}$ is a finite set. If $\mathcal{U}(\mathcal{H})$ is l.c. $\exists$ cpt C s.t. $U_{F,\varepsilon} \subset C$.

Write $\mathcal{H} = \langle F \rangle \oplus \langle F \rangle^\perp$

Obviously the sbgp

$$\begin{pmatrix} Id & 0 \\ 0 & \mathcal{U}(\langle F\rangle^\perp) \end{pmatrix} \subset U_{F,\varepsilon}$$

In fact if $T \in \begin{pmatrix} Id & 0 \\ 0 & \mathcal{U}(\langle F\rangle^\perp) \end{pmatrix}$

$\Rightarrow T = Id \oplus T'$ and

$Tu = u \quad \forall u \in F \Rightarrow$

$\Rightarrow T \in U_{F,\varepsilon}$

But $\begin{pmatrix} Id & 0 \\ 0 & \mathcal{U}(\langle F\rangle^\perp) \end{pmatrix} \simeq \mathcal{U}(\langle F\rangle^\perp)$

$\Rightarrow \mathcal{U}(\langle F\rangle^\perp) \subset U_{F,\varepsilon} \subset C$

$\Rightarrow \boxed{\mathcal{U}(\langle F\rangle^\perp) \subset C}$

Since F is finite, if $\mathcal{H}$ is inf. dim $\Rightarrow$ $\dim \langle F\rangle^\perp = \infty$

Claim $\mathcal{U}(\infty.\text{dim. Hilbert space})$ cannot be contained in a compact set.

Let us suppose that we can find a sequence

$(T_n) \subset \mathcal{U}(\mathcal{H})$ of unitary operators converging to zero in the weak operator topology.

We said that the WOT = SOT on $\mathcal{U}(\mathcal{H}) \Rightarrow (T_n) \to 0$ in the SOT as well.

But this is impossible since $\mathcal{U}(\mathcal{H})$ is closed in the SOT $\Rightarrow \lim T_n$ is also unitary. Hence $|\det(\lim T_n)| = 1$ and $\det 0 = 0$.

Now we need to find such a sequence of operators.

$\mathcal{H}$ ∞-dim $\Rightarrow$ $\boxed{\mathcal{H} \simeq L^2(\mathbb{R})}$

Let $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$

$f \mapsto f(x-1)$

$(Tf)(x) = f(x-1)$.

T^n, Recall that C^∞ cpt. supported fcts are dense in $L^2(\mathbb{R})$, f, g such functions $\Rightarrow$

$\langle T^n f, g \rangle \xrightarrow[n\to\infty]{} 0$