$$A_{det} = \left\{ \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} : \lambda_j \in \mathbb{R}^\times \right\}$$

$$N = \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} : * \text{ are all in } \mathbb{R} \right\}$$

$$O(n,\mathbb{R}) = \{ X \in GL(n,\mathbb{R}) : X^t X = 1 \}$$

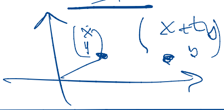
$$SL(n,\mathbb{R}) = \{ X \in GL(n,\mathbb{R}) : \det X = 1 \}$$

- $A_{det} \cap SL(n,\mathbb{R}) =: A$ (dilations)

$\underline{n=2}$ $\quad \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda \\ 0 \end{pmatrix}$

- $N < SL(n,\mathbb{R})$ (shearings)

$\underline{n=2}$ $\quad \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+ty \\ y \end{pmatrix}$



$$SO(n,\mathbb{R}) := O(n,\mathbb{R}) \cap SL(n,\mathbb{R})$$
(rotations)

- Properties of topological gps
- Haar measure

Proposition G top. gp.

(1) $H < G$ subgp $\Rightarrow \bar{H}$ sbgp.

(2) Every discrete & normal sbgp is central (G connected)

(3) the connected component G^0 of the identity is a closed sbgp.

(4) Every open sbgp is closed

Rk Not every closed sbgp is open, e.g. $\mathbb{R} < \mathbb{R}^2$ is closed but not open.

Pf (1) $R_g : G \to G$, $x \mapsto xg$, $g \in G$

is cont. since G is a top. gp, in fact a homeo $(R_g)^{-1} = R_{g^{-1}}$

If $h \in H \Rightarrow R_h H \subset H$ & by continuity $R_h \bar{H} \subset \bar{H}$. Moreover if $h_n \to g \in \bar{H} \Rightarrow$ by cont. $R_{h_n} x \to R_g x$ $\forall x \in G$. Need to see that if $x \in \bar{H} \Rightarrow R_g x \in \bar{H}$. But this is true since $\bar{H}$ is closed.

(2) Let D be a discrete normal sbgp $D < G$. Fix $h \in D$ and define

$c_h : G \to D$ (normal)

$g \mapsto ghg^{-1}$

Goal: Show that $ghg^{-1} = h$

G connected $\Rightarrow c_h(G) = pt$ (discrete)

$c_h(e) = ehe = h \Rightarrow ghg^{-1} = h$ $\forall g \in G$.

(3) $m : G^0 \times G^0 \to G$ but $G^0 \times G^0$ conn. $\Rightarrow m(G^0 \times G^0)$ conn. $\Rightarrow m(G^0 \times G^0) \subset G^0$ i.e. G^0 is closed under multiplication. Likewise $i : G^0 \to G \Rightarrow i(G^0) \subset G^0$.

$G^0 \subset \overline{G^0} \subset G^0 \Rightarrow G^0$ closed

(4) $H < G$ open sbgp. $L_g : G \to G \Rightarrow L_g H$ is also open

$$\bigcup_{\substack{g \neq e \\ g \notin H}} L_g H = \underbrace{G \setminus H}_{\text{open}}$$

∎

Haar measure

X l.c. top. space
G top. grp.
A ~~left~~ right action of G on X is a homo $\phi_\ell : G \longrightarrow \mathrm{Homeo}(X)$ by homeo

$$\phi_\ell(gh) = \phi_\ell(g)\,\phi_\ell(h),$$

that is a map $G \times X \to X$
$(g,x) \mapsto gx$
such that $(g_1 g_2, x) = (g_2, (g_1, x))$

$(g_1 g_2, x) = (g_1, (g_2, x))$

The action is continuous if the map $G \times X \to X$ is cont. in which case

$\varphi_g : X \to X$ is a homeo
$x \mapsto gx$

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whose inverse is $\varphi_{g^{-1}}$

If $C_c(X)$ = cont. f. on X with cpt support $\Rightarrow$

$\Rightarrow$ G acts cont. on $C_c(X)$,

$$(\lambda(g)f)(x) := f(g^{-1}x).$$

Likewise, G acts on $C_c(X)^*$ via the adjoint action

$$(\lambda^*(g)(\Lambda))(f) := \Lambda(\lambda(g^{-1})f)$$

$$\Lambda \in C_c(X)^*, \quad f \in C_c(X)$$

Thm (Riesz repr. thm)
X l.c. Hausdorff top. space, Λ positive linear functional on $C_c(X)$ ($\Lambda f \geq 0$ $\forall f \in C_c(X)$ $f \geq 0$).

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$\Rightarrow$ $\exists$ regular Borel measure μ on X that represents Λ, that

$$\Lambda(f) = \int_X f(x)\, d\mu(x)$$

Recall (1) A Borel measure μ on X is a m on $\mathcal{B}(X)$ which is finite on compact sets.

(2) $\forall$ Borel set Y,

$$\mu(Y) = \sup_{\substack{K \text{ cpt} \\ K \subset Y}} \mu(K) \quad \text{(inner reg.)}$$

$$\mu(Y) = \inf_{\substack{U \text{ open } \sigma\text{-bounded} \\ U \supset Y}} \mu(U)$$

(σ-bounded means contained in a countable union of cpt. sets)

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$$\boxed{(\lambda(g)f)(x) := f(g^{-1}x)}$$

reg. rep. are $G \to \mathrm{Iso}(L^2(G))$

$\lambda : G \to \mathrm{Iso}(L^2(G))$

$(\lambda(g)f)(x) := f(g^{-1}x)$

$\rho : G \to \mathrm{Iso}(L^2(G))$

$(\rho(g)f)(x) := f(xg)$

$f \in L^2(G)$.

$G \curvearrowright X \rightsquigarrow G \curvearrowright C_c(X) \rightsquigarrow$
$\rightsquigarrow G \curvearrowright (C_c(X))^* \rightsquigarrow$
$\rightsquigarrow G \curvearrowright$ regular Borel m

$$(g_*\mu)(A) := \mu(g^{-1}A)$$

A Borel set.

$$(g\Lambda)(f) = \int_X f(x)\, d g_*\mu(x) =$$

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$$= \int_X f(x)\, d\mu(g^{-1}x)$$

Defn. A left (resp right) Haar measure on a l.c. Hausdorff G is a non-zero positive linear functional
$$m: C_c(G) \to \mathbb{C},$$
that is invariant under left (resp. right) translation
$$(g_* m)(f) = m(f)$$
$\forall f \in C_c(G)$.

Notation: $m(f)$, $\int_G f(x)\,dm(x)$, $dm(x)$, dx

Thm (1933) A left (resp. right) Haar measure on a l.c. Hausdorff gp exists and is unique up to positive multiplicative constants.

Lemma: Let m be a left Haar measure and let $\check{f}(x) := f(x^{-1})$. Then $n(f) := m(\check{f})$ is a right Haar measure.

Pf: Need to show that $n(\rho(g)f) = n(f)$ $\forall g \in G$, $f \in C_c(G)$.

$$(\rho(g)f)^\vee(x) = (\rho(g)f)(x^{-1}) =$$
$$= f(x^{-1}g)$$

$$n(\rho(g)f) = m((\rho(g)f)^\vee) =$$
$$= \int_G f(x^{-1}g)\,dm(x) =$$
$$= \int_G \check{f}(g^{-1}x)\,dm(x)$$
$$= \int_G \check{f}(x)\,dm(x) = n(f) \quad \blacksquare$$

Lemma: G l.c. Hausdorff with left Haar measure m. Then (1) $\mathrm{supp}\, m = G$

(2) If $h \in C_c(G)$ s.t.
$$\int_G h(x)\varphi(x)\,dm(x) = 0$$
$\forall \varphi \in C_c(G)$, then $h \equiv 0$.

(1) $\mathrm{supp}(m) := \{x \in G :$ for every open set $U \ni x$ $m(U) > 0\}$.

Since $m \not\equiv 0 \Rightarrow \exists f \in C_c(G)$ s.t. $m(f) > 0$. Let $K = \mathrm{supp}(f)$ with $m(K) > 0$.

If $G \neq \mathrm{supp}(m) \Rightarrow \exists$ $x \in G \setminus \mathrm{supp}(m)$ and $\exists$ open nbd $U \ni x$ s.t. $m(U) = 0$. But then one could cover K with translates of U and in fact with finitely many translates of U, which is a contradiction.

($U \ni x \Rightarrow x^{-1}U \ni e$, $\{kx^{-1}U\}_{k \in K}$ is an open cover of K)

(2) We show that $h(e)=0$. Let $\varepsilon>0 \Rightarrow \exists$ open nbd $V\ni e$ s.t.

$$|h(g)-h(e)|<\varepsilon \quad \forall g\in V.$$

Urysohn's lemma $\Rightarrow \exists$ $\varphi\in C_c(G)$ s.t.

$\varphi\geq 0$, $\varphi(e)>0$ and $\operatorname{supp}\varphi\in V$. Recall that

$$\int_G h(g)\varphi(g)\,dm(g)=0,$$

so that

$$=|h(e)|\left|\int_G \varphi(g)\,dm(g)\right|$$

$$=\left|\int_G h(e)\varphi(g)\,dm(g)\right|$$

$$=\left|\int_G h(g)\varphi(g)\,dm(g)-\int_G h(e)\varphi(g)\,dm(g)\right|$$

$$=\left|\int_G |h(g)-h(e)|\varphi(g)\,dm(g)\right|$$

$$<\varepsilon\left|\int_G \varphi(g)\,dm(g)\right|$$

$$\Rightarrow |h(e)|<\varepsilon \quad \forall\,\varepsilon>0$$

$$\Rightarrow h(e)=0.\ \blacksquare$$