Ex. day 1 for Lie Groups

Fundamental group of a topological group:

Examples:

(a) $\mathbb{R}^n$, $\pi_1(\mathbb{R}^n) = \{0\}$.

(b) S^1, $\pi_1(S^1) = \mathbb{Z}$

$\mathbb{T}^n = \underbrace{S^1 \times \ldots \times S^1}_{n\text{-times}}$,

$\pi_1(\mathbb{T}^n) = \pi_1(S^1) \times \ldots \times \pi_1(S^1)$
$= \mathbb{Z}^n$

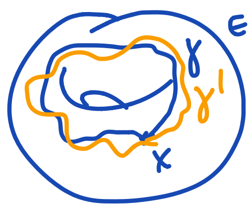
(c) $SO(2,\mathbb{R}) = \{ A \in \mathbb{R}^{2\times 2} \mid A^T A = I_2 \}$

$$A = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}$$

$\rightsquigarrow$ $SO(2,\mathbb{R}) \cong S^1$.
$\pi_1(SO(2,\mathbb{R})) = \mathbb{Z}$

Fundamental gp: X top. space, $x \in X$

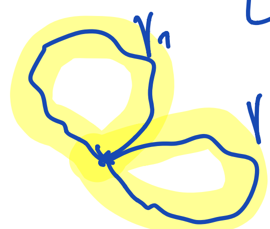
$\Omega(X,x) := \{ \gamma : [0,1] \to X : \gamma(0) = \gamma(1) = x \}$



$\in \Omega(X,x)$.

$\pi_1(X,x) = \Omega(X,x) / \text{htpy rel endpts.}$

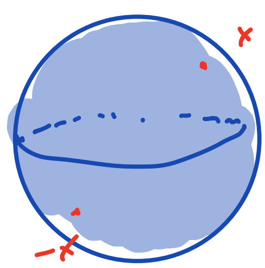
$[\gamma_1], [\gamma_2] \in \pi_1(X,x)$



$[\gamma_1] \cdot [\gamma_2] = [\gamma_1 * \gamma_2]$

$SO(3,\mathbb{R}) \cong \mathbb{RP}^3 \cong S^3 / (\mathbb{Z}/2\mathbb{Z}) \Longrightarrow \pi_1(SO(3,\mathbb{R}))$
$= \pi_1(\mathbb{RP}^3) = \mathbb{Z}/2\mathbb{Z}$.

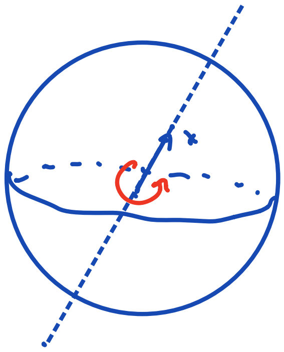
$\mathbb{RP}^3 = \mathbb{D}^3 / _{x \sim -x \text{ if } \|x\| = 1}$

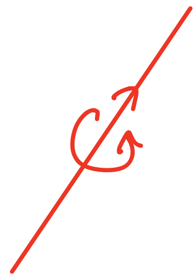


$\mathbb{D}^3 = \{ x \in \mathbb{R}^3 \mid \|x\| \leq 1 \}$

$$\varphi : \mathbb{D}^3 \longrightarrow SO(3,\mathbb{R})$$
$$x \longmapsto \varphi(x) = \text{rotation about axis } \mathbb{R}\cdot x$$

by $\pi \cdot \|x\|$ degrees "to the left" (defined by "the right-hand rule")





Check: φ is surjective & induces a homeom. $\overline{\varphi}$ on the quotient:

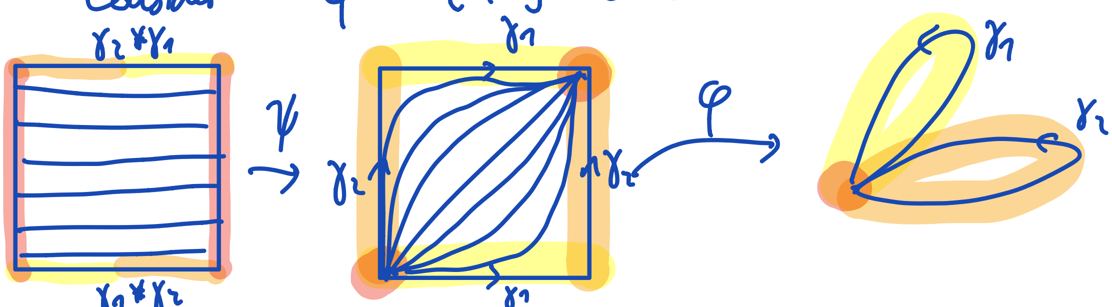
$$\begin{tikzcd} \mathbb{D}^3 \arrow[r, "\varphi"] \arrow[d, "\pi"] & SO(3,\mathbb{R}) \\ \mathbb{RP}^3 = \mathbb{D}^3/\sim \arrow[ur, dotted, "\overline{\varphi}", "\cong"'] & \end{tikzcd}$$

Observation: All of them are abelian!

Prop: Let G be a topological group. Then $\pi_1(G)$ is abelian.

Proof: Let $[\gamma_1], [\gamma_2] \in \pi_1(G,e)$.

Consider $\varphi : [0,1]\times[0,1] \longrightarrow G$, $\varphi(t,s) := \gamma_1(t)\gamma_2(s)$



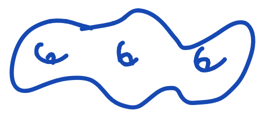
We obtain a homotopy $H := \varphi \circ \psi$ between $\gamma_1 * \gamma_2$ and $\gamma_2 * \gamma_1$.

$$\Longrightarrow \quad [\gamma_1] \cdot [\gamma_2] = [\gamma_1 * \gamma_2] \overset{H}{=} [\gamma_2 * \gamma_1] = [\gamma_2] \cdot [\gamma_1] \qquad \forall [\gamma_1], [\gamma_2] \in \pi_1(G, e)$$

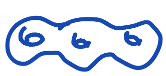
$\Longrightarrow \quad \pi_1(G, e)$ is abelian. $\square$

Rmk: This result extends to H-spaces X that only admit a "sort of" multiplication map $\mu : X \times X \to X$.

Application:

Question: Does a closed surface $\Sigma =$  admit a topological group structure?

Answer: No!

$\pi_1\Big($  $\Big) = \langle a_1, b_1, a_2, b_2, a_3, b_3 \mid \prod_{i=1}^{3} [a_i, b_i] = 1 \rangle$

is NOT abelian!

Universal covering group: (locally path-conn, semiloc. simply conn.)

Let G be a path-connected top. gp with a universal covering $\tilde{G}$.

$\tilde{G}$: only a topological space so far.

Prop: The universal covering $\tilde{G}$ admits a (top.) group structure such that $\pi: \tilde{G} \to G$ is a (top.) group homomorphism.

Proof: Denote $m: G \times G \to G$, $(g,h) \mapsto g \cdot h$, $\quad i: G \to G$, $g \mapsto g^{-1}$.

Pick $\tilde{e} \in \pi^{-1}(e)$. This will be our identity element.

$$(*) \quad \begin{tikzcd} \tilde{G} \times \tilde{G} \arrow[r, dashed, "\tilde{m}"] \arrow[d, "\pi \times \pi"'] \arrow[dr, phantom, "="] & \tilde{G} \arrow[d, "\pi"] \\ G \times G \arrow[r, "m"] & G \end{tikzcd}$$

$\pi \times \pi$ is a covering map.

We lift m to $\tilde{m}$ s.t. $\tilde{m}(\tilde{e}, \tilde{e}) = \tilde{e}$.

π is a group homomorphism.

$$\begin{tikzcd} \tilde{G} \arrow[r, dashed, "\tilde{i}"] \arrow[d, "\pi"'] \arrow[dr, phantom, "="] & \tilde{G} \arrow[d, "\pi"] \\ G \arrow[r, "i"] & G \end{tikzcd}$$

Lift i to $\tilde{i}$ s.t. $\tilde{i}(\tilde{e}) = \tilde{e}$.

One can check that $\tilde{G}$ is a top. gp. with mult. $\tilde{m}$ and inversion $\tilde{i}$.

E.g.: $\tilde{i}$ gives inverses for $\tilde{m}$:

WTS: $\tilde{m}(\tilde{g}, \tilde{i}(\tilde{g})) \overset{?}{=} \tilde{e} \quad \forall \tilde{g} \in \tilde{G}$

$$\tilde{m} \circ (id \times \tilde{i}) \equiv \tilde{e} \quad (\text{const.})$$

$$\begin{tikzcd}
\tilde{G} \times \tilde{G} \arrow[r, "id \times \tilde{i}"] \arrow[d, "\pi \times \pi"] \arrow[dr, phantom, "//"] & \tilde{G} \times \tilde{G} \arrow[r, "\tilde{m}"] \arrow[d, "\pi \times \pi"] \arrow[dr, phantom, "//"] & \tilde{G} \arrow[d, "\pi"] \\
G \times G \arrow[r, "id \times i"] \arrow[rr, bend right, dashed, "e"'] & G \times G \arrow[r, "m"] & G
\end{tikzcd}$$

Note that $\tilde{m} \circ (id \times \tilde{i})$ is a lift of $m \circ (id \times i) \equiv e$

But so is $\tilde{e}$. By uniqueness of lifts:

$$\tilde{m} \circ (id \times \tilde{i}) \equiv \tilde{e} \quad \checkmark$$

Other statements follow similarly (Exercise).

Rmk: This group structure is unique:

Let $\tilde{G}_1, \tilde{G}_2$ be two simply conn. top. gps covering G, s.t. $\pi_1 : \tilde{G}_1 \to G$, $\pi_2 : \tilde{G}_2 \to G$ ~~are~~ are homomorphisms. Then there is a top. isomorphism φ s.t.

$$\begin{tikzcd}
\tilde{G}_1 \arrow[rr, "\varphi"] \arrow[dr, "\pi_1"'] & \arrow[d, phantom, "//"] & \tilde{G}_2 \arrow[dl, "\pi_2"] \\
& G &
\end{tikzcd}$$

Consider $\Gamma := \ker(\pi)$ for the universal covering group $\pi : \tilde{G} \longrightarrow G$.

<u>Prop:</u> Γ is discrete and normal. In particular, Γ is central.

<u>Proof:</u>

- <u>Γ is discrete:</u> π is a covering map, whence there are open nbhds. $V \subseteq \tilde{G}$, $U \subseteq G$ of $\tilde{e}, e$ resp. s.t. $\pi|_V : V \xrightarrow{\cong} U$.

  Then $\Gamma \cap \gamma \cdot V \underset{\subseteq ?}{\overset{?}{=}} \{\gamma\}$ $\forall \gamma \in \Gamma$.

- $\subseteq$: If $\gamma_1, \gamma_2 = \gamma_1 \cdot v \in \Gamma \cap \gamma \cdot V$ (with $v \in V$) then

  $$e = \pi(\gamma_2) = \pi(\gamma_1 \cdot v) = \underbrace{\pi(\gamma_1)}_{= e} \cdot \pi(v) = \pi(v)$$

  $\overset{\pi|_V \text{ inj.}}{\Longrightarrow}$ $v = \tilde{e}$.

- <u>Γ central:</u> $\Gamma \trianglelefteq \tilde{G}$ & discrete.

  $\Longrightarrow$ Γ is central: $\gamma \cdot \tilde{g} = \tilde{g} \cdot \gamma$ $\forall \gamma \in \Gamma$ $\forall \tilde{g} \in \tilde{G}$.

  Let $\tilde{g} \in \tilde{G}$, $\gamma \in \Gamma$, and let $\tilde{c} : [0,1] \longrightarrow \tilde{G}$ be a cont. path from $\tilde{e} = \tilde{c}(0)$ to $\tilde{g} = \tilde{c}(1)$.

  Then $\tilde{c}(t) \cdot \gamma \cdot \tilde{c}(t)^{-1} \in \Gamma$ $\forall t \in [0,1]$.

  $t \longmapsto \tilde{c}(t) \cdot \gamma \cdot \tilde{c}(t)^{-1}$ is cont. with discrete image

  $\Longrightarrow$ it's constant.

$$\tilde{g} \cdot \gamma \cdot \tilde{g}^{-1} = \tilde{c}(1) \cdot \gamma \cdot \tilde{c}(1)^{-1} = \tilde{c}(0) \cdot \gamma \cdot \tilde{c}(0)^{-1}$$
$$= \tilde{e} \cdot \gamma \cdot \tilde{e}^{-1} = \gamma$$

$\square$

Rmk: In fact, Γ can be identified with the group of deck transformations and is hence isomorphic to $\pi_1(G)$:

$$\Gamma \cong \mathrm{Deck}(\pi) \cong \pi_1(G).$$