Examples:

(a) \( \mathbb{R}^n, \pi_n(\mathbb{R}^n) = \{0\} \)

(b) \( S^1, \pi_n(S^1) = \mathbb{Z} \)

\[ T^n = S^1 \times \cdots \times S^1 \text{ (n-times)} \]

\[ \pi_n(T^n) = \pi_n(S^1) \times \cdots \times \pi_n(S^1) = \mathbb{Z}^n \]

(c) \( \text{SO}(2, \mathbb{R}) = \{ A \in \mathbb{R}^{2 \times 2} \mid A^T A = I_2 \} \)

\[ A = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \]

\( \Rightarrow \text{SO}(2, \mathbb{R}) \cong S^1 \)

\[ \pi_1(\text{SO}(2, \mathbb{R})) = \mathbb{Z} \]

\[ \text{SO}(3, \mathbb{R}) \cong \mathbb{R}P^3 \cong S^3 / \mathbb{Z}/2 \mathbb{Z} \Rightarrow \pi_n(\text{SO}(3, \mathbb{R})) = \pi_n(\mathbb{R}P^3) = \mathbb{Z}/2 \mathbb{Z} \]

\[ \mathbb{R}P^3 = D^3 / \sim \text{ if } \|x\| = 1 \}

\[ D^3 = \{ x \in \mathbb{R}^3 \mid \|x\| \leq 1 \} \]
\( \varphi: D^3 \rightarrow SO(3, \mathbb{R}) \)

\( \varphi(x) = \text{rotation about axis } P_x \) by \( \pi \times \pi \) degrees "to the left" (defined by "the right-hand rule")

\( D^3 \) 
\[ D^3 \xrightarrow{\varphi} SO(3, \mathbb{R}) \]

\( \mathbb{R}P^3 = D^3 / \sim \)

**Claim:** \( \varphi \) is surjective & induces a homeomorphism \( \overline{\varphi} \) on the quotient:

**Observation:** All of them are **abelian**!

**Prop:** Let \( G \) be a topological group.

Then \( \pi_1(G) \) is **abelian**.

**Proof:** Let \( [\gamma_1], [\gamma_2] \in \pi_1(G, e) \).

Consider \( \varphi: [0, 1] \times [0, 1] \rightarrow G \), \( \varphi(0, s) = \gamma_1(c) \gamma_2(s) \).
We obtain a homotopy \( H = \varphi \circ \psi \) between \( Y_0 \ast Y_1 \) and 
\( Y_2 \ast Y_3 \).

\[
\left[ [a] \ast [f_2] \right] = \left[ [g_3] \ast [g_4] \right] = \left[ [g_5] \cdot [g_6] \right] = [a] \cdot [f] \cdot [g] \cdot [h]
\]

\( \therefore \pi_7(G, e) \) is abelian.

The result extends to \( H \)-spaces \( X \) that only 
admit a "sort of" multiplication map \( \mu : X \times X \to X \).

Application:

**Question:** Does a closed surface \( \Sigma = \) admit a topological group structure?

**Answer:** No!

\[
\pi_3 \left( \begin{array}{cc}
\includegraphics[width=2cm]{closed_surface}
\end{array} \right) = \langle a_1, b_1, a_2, b_2, a_3, b_3 \mid \prod_{i=1}^{3} [a_i, b_i] = 1 \rangle
\]

is NOT abelian!
Universal covering group: (locally path conn, semiloc. simply conn.)

Let \( \tilde{G} \) be a path-connected top. gp. with a universal covering \( \tilde{G} \).

\( \tilde{G} \) only a topological space so far.

Prop: The universal covering \( \tilde{G} \) admits a group structure such that \( \tilde{\pi}: \tilde{G} \rightarrow \tilde{G} \) is a group homomorphism.

Proof: Denote \( m: \tilde{G} \times \tilde{G} \rightarrow \tilde{G} \), \( i: \tilde{G} \rightarrow \tilde{G} \)

\[ (g, h) \mapsto g \cdot h, \quad g \mapsto \tilde{g} \cdot \tilde{e}^i. \]

Pick \( \tilde{e} \equiv i(\tilde{e}) \). This will be our identity element.

\[ \tilde{G} \times \tilde{G} \xrightarrow{m} \tilde{G} \]

It is a group homomorphism.

\[ \tilde{\pi} \times \tilde{\pi} \] is a covering map.

We lift \( m \) to \( \tilde{m} \) s.t. \( \tilde{m}(\tilde{e}, \tilde{e}) = \tilde{e} \).

Lift \( i \) to \( \tilde{i} \) s.t.

\[ i(\tilde{e}) = \tilde{e}. \]

One can check that \( \tilde{G} \) is a top. gp. with mult. \( \tilde{m} \) and inversion \( \tilde{i} \).
E.g.: \( \bar{g} \) gives inverses for \( \bar{g} \):

\[
\text{w.t.s.: } \bar{m} (\bar{g}, \bar{i}(\bar{g})) = \bar{e} \quad \forall \bar{g} \in \bar{G}
\]

\[
\bar{m} \circ (\text{id} \times \bar{i}) = \bar{e} \quad \text{(const.)}
\]

\[
\begin{array}{ccc}
\bar{G} \times \bar{G} & \xrightarrow{id \times \bar{i}} & \bar{G} \times \bar{G} & \xrightarrow{\bar{m}} & \bar{G} \\
\downarrow \pi \times \pi & & \downarrow \pi \times \pi & & \downarrow \pi \\
G \times G & \xrightarrow{id \times i} & G \times G & \xrightarrow{m} & G
\end{array}
\]

Note that \( \bar{m} \circ (\text{id} \times \bar{i}) \) is a lift of \( m(\text{id} \times \bar{i}) = e \).

But so is \( \bar{e} \). By uniqueness of lifts:

\[
\bar{m} \circ (\text{id} \times \bar{i}) = \bar{e} \quad \checkmark
\]

Other statements follow similarly (Exercise).

\textbf{Remark:} This group structure is unique:

Let \( \tilde{G}_1, \tilde{G}_2 \) be two simply count. top. gps covering \( G \), s.t. \( \pi_1 : \tilde{G}_1 \rightarrow G, \pi_2 : \tilde{G}_2 \rightarrow G \) are homomorphisms. Then there is a top. isomorphism of \( \tilde{G}_1 \) s.t. \( \tilde{G}_1 \rightarrow \tilde{G}_2 \)
Consider $\Gamma := \ker(\pi)$ for the universal covering group $\pi: \tilde{G} \to G$.

**Prop:** $\Gamma$ is discrete and normal. In particular, $\Gamma$ is central.

**Proof:**

- **$\Gamma$ is discrete:** $\pi$ is a covering map, whence there are open nbhds $V \subseteq \tilde{G}$, $U \subseteq G$ of $\tilde{e}, e$ resp. s.t. $\pi|_V : V \xrightarrow{\sim} U$.

  Then $\Gamma \cap \pi^{-1}(U) \subseteq \{e\} \quad \forall \gamma \in \Gamma$.

  - If $\gamma_1, \gamma_2 \in \pi^{-1}(U)$, then
    
    $e = \pi(\gamma_1) = \pi(\gamma_1 \cdot \gamma_2 \cdot \gamma_1^{-1}) = \pi(\gamma_1 \cdot \gamma_2) \cdot \gamma_1^{-1} \cdot \pi(\gamma_1) = \pi(\gamma_2) = \pi(\gamma_2)$

    $\pi|_V$ inj.

    $\implies \quad \gamma_2 \cdot \gamma_1^{-1} \in \Gamma$.

- **$\Gamma$ central:** $\Gamma \triangleleft \tilde{G}$ is discrete.

  $\implies \quad \Gamma$ is central: $\gamma \cdot \tilde{g} = \tilde{g} \cdot \gamma \quad \forall \gamma \in \Gamma, \tilde{g} \in \tilde{G}.$

  Let $\tilde{g} \in \tilde{G}$, $\gamma \in \Gamma$, and let $\tilde{c}: [0,1] \to \tilde{G}$ be a cont. path from $\tilde{e} = \tilde{c}(0)$ to $\tilde{g} = \tilde{c}(1)$.

  Then $\tilde{c}(t) \cdot \gamma \cdot \tilde{c}(t)^{-1} \in \Gamma \quad \forall t \in [0,1]$.

  $t \mapsto \tilde{c}(t) \cdot \gamma \cdot \tilde{c}(t)^{-1}$ is cont. with discrete image

  $\implies$ it's constant.
\[
\tilde{g} \cdot \gamma \cdot \tilde{g}^{-1} = \tilde{c}(g) \cdot \gamma \cdot \tilde{c}(g)^{-1} = \tilde{c}(g) \cdot \gamma \cdot \tilde{c}(g)^{-1} = \tilde{c} \cdot \gamma \cdot \tilde{c}^{-1} = \gamma
\]

Puki: In fact, \( \Gamma \) can be identified with the group of deck transformations and is hence isomorphic to \( \pi_1(S) \):
\[
\Gamma \cong \text{Deck}(S) \cong \pi_1(S).
\]