Recall A left Haar measure on a locally compact Hausdorff topol. gp. is a non-zero positive linear functional $m: C_c(G) \to \mathbb{R}$ that is invariant for left translations
$$m(\lambda(g)f) = m(f)$$
$\forall f \in C_c(G)$, $g \in G$.

Recall $G \curvearrowright X$ cont. on the left
$\Rightarrow G \curvearrowright C_c(X)$ cont. on the left, i.e. $\lambda: G \to \mathrm{Iso}(C_c(X))$ ($\mathrm{Iso}(C_c(X))$ has the s.o.t.)
$\Rightarrow G \curvearrowright C_c(X)^*$ cont. on the left via the adj. repr. (here $C_c(X)^*$ has the w-*-top, i.e.
$$\lambda^*(g)(\Lambda)(f) := \Lambda(\lambda(g)^{-1}f)$$

The weak-*-top. on E^* is the top. for which if $(\Lambda_n) \subset E^*$, then $\Lambda_n \to \Lambda$ iff $\forall f \in E$, $\langle \Lambda_n, f\rangle \to \langle \Lambda, f\rangle$

/1

Last time:

Thm (1933) A left Haar Ⓜ on a l.c. Hausdorff top. gp. $\exists$ and is unique up to positive multiplic. constants.

Proof of uniqueness later

lemma: (1) $\mathrm{supp}(m) = G$

(2) If m is a left Haar Ⓜ, then $n(f) := m(\check{f})$, $\check{f}(x) := f(x^{-1})$, is a right Haar Ⓜ

(3) If $h \in C_c(G)$ has the property that $\int_G h(g)\varphi(g)\,dm(g) = 0$
$\Rightarrow h \equiv 0$

/2

Pf of uniqueness m left Haar Ⓜ, n right Haar Ⓜ, $f, g \in C_c(G)$

$$m(f)n(g) = m(f)\int_G g(y)\,dn(y) =$$
$$= m(f)\int_G g(yt)\,dn(y) =$$
n right-inv.
$$= \int_G f(t)\left(\int_G g(yt)\,dn(y)\right)dm(t)$$
Fubini
$$= \int_G\left(\int_G f(t)g(yt)\,dm(t)\right)dn(y)$$
m left inv. $yt = x$
$$= \int_G\int_G f(y^{-1}x)g(x)\,dm(x)\,dn(y)$$
Fubini
$$= \int_G \underbrace{\left(\int_G f(y^{-1}x)\,dn(y)\right)}_{m(f)\, w_f(x),\ w_f: G \to \mathbb{R}} g(x)\,dm(x)$$

/3

$$\cancel{m(f)}\,n(g) = \cancel{m(f)}\int w_f(x)g(x)\,dm(x)$$
LHS does not depend on $f \Rightarrow$
$$\Rightarrow \int_G (w_{f_1}(x) - w_{f_2}(x))g(x)\,dm(x) = 0$$
$\forall f_1, f_2, g \in C_c(G) \Rightarrow$
$\Rightarrow w_f(x)$ does not dep. on f.
Set $C := w_f(e)$. Thus
$$m(f)C = m(f)w_f(e) =$$
$$= \int_G f(y^{-1})\,dn(y) = n(\check{f})$$
Now if m, m' are two left Haar Ⓜ, set $n(f) := m'(\check{f})$
$$\Rightarrow m(f)\,C = m'(f),$$
$\forall f \in C_c(G)$, C indep. of f. ☒

/4

Examples (1) $G = (\mathbb{R}^n, +) \Rightarrow$ the Lebesgue (m) is both left & right inv.

(2) $(\mathbb{R}_{>0}, \cdot) \Rightarrow$ the Lebesgue (m) is neither left nor right inv.

However $\frac{dx}{x}$ is a left Haar (m). (hence also right inv.)

(3) G discrete gp $\Rightarrow$ the counting (m) is both left & right invariant.

Question When is the left Haar (m) also right inv.?

$\alpha \in Aut(G)$ (cont. & inv.), acts on $C_c(G)$

$(\alpha . f)(x) := f(\alpha^{-1}(x))$

Easy to verify: the linear fund. $f \mapsto m(\alpha . f)$ is a

/5

left Haar integral if m is$^{(*)}$ $\Rightarrow$

$\Rightarrow \exists$ constant $mod_G(\alpha)$ s.t.

$$\boxed{m(\alpha . f) = mod_G(\alpha)\, m(f)}$$

$\forall f \in C_c(G)$

Lemma $mod_G : Aut(G) \to \mathbb{R}$ is a homo, i.e.

$$mod_G(\alpha\beta) = mod_G(\alpha)\, mod_G(\beta)$$

(*) To verify $m((\lambda(g))(\alpha . f)) = m(\alpha . f)$

Pf of lemma: ex

Example $Aut(\mathbb{R}^n) = GL(n, \mathbb{R})$ and $mod_{GL(n,\mathbb{R})} : GL(n,\mathbb{R}) \to \mathbb{R}$ is $mod_{GL(n,\mathbb{R})}(\alpha) = |\det \alpha|^{-1}$.

Particularly relevant autom.

$\alpha := c_g$, $g \in G$,

$c_g(h) := g^{-1} h g \Rightarrow$

$(c_g f)(x) = f(g x g^{-1})$

/6

Notation $mod_G(c_g) =: \Delta_G(g)$ is called the modular function of G.

$$\Rightarrow \boxed{m(c_g f) = \Delta_G(g)\, m(f)}$$

and Δ_G captures how far m is from being right inv.

$$m(c_g f) = \int_G (c_g f)(x)\, dm(x) = \int_G f(g x g^{-1})\, dm(x) = \int_G f(x g^{-1})\, dm(x)$$

$$\Delta_G(g)\, m(f) = \Delta_G(g) \int_G f(x)\, dm(x)$$

/7

$$\Rightarrow \Delta_G(g) \int_G f(xg)\, dm(x) = \int_G f(x)\, dm(x)$$

$$\Rightarrow \Delta_G(g)\, m(\rho(g) f) = m(f)$$

how far m is from being right inv. that $m(\rho(g)f) = m(f)$

Facts about top. gps

(1) Every nbd $U \ni e$ contains a symmetric nbd $W = W^{-1}$ (take $W = U \cap U^{-1}$).

/8

(2) Every nbd $U \ni e$ contains a nbd $V \ni e$ s.t. $V = V^{-1}$

$$V \cdot V = V \cdot V^{-1} \subset U.$$

(Continuity of multiplication)

Proposition (1) Δ_G is continuous

(2) $\forall f \in C_c(G)$

$$\int_G f(x^{-1})\Delta_G(x^{-1})\,dm(x) = \int_G f(x)\,dm(x)$$

Pf (1) $\rho : G \to \mathrm{Iso}(C_c(X))$ cont. in the strong op. top.

$\Rightarrow \lim_{x \to y} \|\rho(x)f - \rho(y)f\|_\infty = 0$

$\forall f \in C_c(G) \Rightarrow$

$\Rightarrow 0 = \lim_{x\to y} |m(\rho(x)f) - m(\rho(y)f)|$

$= \lim_{x \to y} |\Delta_G(x)\,m(f) - \Delta_G(y)\,m(f)|$

$= |m(f)| \lim_{x\to y} |\Delta_G(x) - \Delta_G(y)|$

(2) Define $f^*(x) := f(x^{-1})\Delta_G(x^{-1})$

verif. $\Rightarrow (\lambda(g)f)^*(x) \overset{(*)}{=} \Delta_G(g)(\rho(g)f^*)(x)$

Claim $m(f^*)$ is a left Haar integral.

Pf Use $(*)$ to show that $m(\lambda(g)f^*) = m(f^*)$.

We write $m'(f) := m(f^*)$

$\Rightarrow \exists\, c > 0$ s.t. $m'(f) = c\,m(f)$.

$$m'(f) = m(f^*) = \int_G f(x^{-1})\Delta_G(x^{-1})\,dm(x)$$

$$c\,m(f) = c\int_G f(x)\,dm(x)$$

Need to show that $c = 1$.

Since Δ_G is cont. $\forall \varepsilon > 0$ $\exists\, V \ni e$, V symm, s.t.

$|\Delta_G(x) - 1| < \varepsilon \quad \forall x \in V.$

Let $f \in C_c(G)$ be a

- symmetric function,
- $\mathrm{supp}(f) \subset V$
- $m(f) = 1$

$|1 - c| = |(1-c)\,m(f)| =$

$= |m(f) - c\,m(f)| \overset{m' \text{ left H.(m)}}{=}$

$= |m(f) - m'(f)| \overset{\text{def of } m'}{=}$

$= |m(f) - m(f^*)| =$

$= |m(f) - m(f\Delta_G^{-1})|$

$= |m((1 - \Delta_G^{-1})f)| <$

$< \varepsilon\,|m(f)| = \varepsilon \Rightarrow c = 1$ ∎

Definition A group G is unimodular if $\Delta_G \equiv 1$, that is if the left Haar (m) is also right invariant

$$\boxed{\Delta_G(g)\,m(\rho(g)f) = m(f)}$$

Examples

(1) Any l.c. Hausdorff Abel. gp is unimodular

(2) Any discrete gp is unim.

(3) Any compact gp is unim.

(4) $GL(n,\mathbb{R})$ is unimodular with $dm(X) = \prod_{i,j=1}^{n} dX_{ij}$, where $X = (X_{ij})_{i,j=1}^{n}$

Claim $|\det(X)|^{-n} dm(X)$ is left inv. and right inv.

(5) $\mathbb{R}_{>0} \times_\eta \mathbb{R}$

$\eta : \mathbb{R}_{>0} \longrightarrow \mathrm{Aut}(\mathbb{R})$

$a \longrightarrow \eta(a)\{b \mapsto ab\}$

$\Rightarrow$ $\mathbb{R}_{>0} \times_\eta \mathbb{R}$ is a gp with product

$(a,b)(a',b') = (aa', b+ab')$

In general, G_1, G_2 top. gps, $\eta : G_1 \longrightarrow \mathrm{Aut}(G_2)$ homo $\Rightarrow$ $G_1 \times G_2$ can be given the structure of semidirect product with multipl.

$(g_1,g_2)(h_1,h_2) =$

$= (g_1 h_1, g_2\, \eta(g_1) h_2)$

$\Rightarrow$ $G_1 \times_\eta G_2$ is a top. gp. with the product topology.

$\mathbb{R}_{>0} \times_\eta \mathbb{R}$ is the gp. of affine transf. of $\mathbb{R}$

$(a,b)x = ax+b$

and can be identified with $\left\{\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a>0,\ b\in\mathbb{R}\right\}$ acting on $\mathbb{R} \subset \mathbb{R}^2$

Claim $m(a,b) = \frac{da}{a^2}\, db$ is a left Haar m but not a right Haar m.

(6) $\left\{\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x,y,z\in\mathbb{R}\right\}$

Heisenberg gp

$\cong \mathbb{R} \times_\eta \mathbb{R}^2$, $\eta : \mathbb{R} \to \mathrm{Aut}(\mathbb{R}^2)$

$\eta(x)\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} y \\ z+xy \end{pmatrix}$

Claim The Heisenberg gp is unimodular with the Lebesgue m

(7) $P = \left\{\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a\neq 0,\ b\in\mathbb{R}\right\}$

$\leq SL(2,\mathbb{R})$

Claim $\frac{da}{a^2}\, db$ is the left Haar m and $da\, db$ is the right Haar m.

Proposition G l.c. Hausdorff, $H \trianglelefteq G$ closed normal sbgp. Then $\Delta_G|_H = \Delta_H$. Thus if G is unim. $\Rightarrow$ H is unim.