So far

- Defn. of left Haar m & right Haar m
- Modular function $\Delta_G : G \to \mathbb{R}$
  - continuous homo
  - $\Delta_G \equiv 1 \iff$ left Haar m is also right inv.
    $\iff$ Haar m is inversion inv.
- Examples

Proposition G l.c. Hausdorff. Then $m(G) < \infty \iff G$ is compact.

Usually $m(K) = 1$ if possible

Proof $(\Leftarrow)$ G cpt $\Rightarrow 1 \in C_c(G)$ & $m(G) = m(1) < \infty$.

$(\Rightarrow)$ If G is not cpt $\Rightarrow$ we cover G with ∞ no. of translates

of disjoint sets of positive m. Let $V \ni e$ a nbd of e with compact closure. G not cpt $\Rightarrow$ cannot cover V with finit. many translates of V. $\Rightarrow$ Construct $(x_n) \subset G$ as follows

$x_1 = e$

$x_2 \notin V$

$x_3 \notin \bigcup_{j=1}^{2} x_j V$

$x_n \notin \bigcup_{j=1}^{n-1} x_j V$

$\exists\ U = U^{-1} \ni e$ such that $U^2 \subset V$

$\Rightarrow x_n \notin \bigcup_{j=1}^{n-1} x_j U^2$.

We claim that

$x_n U \cap x_k U = \emptyset$ if $n \neq k$.

Set $n > k$. If $x \in x_n U \cap x_k U$

$\Rightarrow x = x_n u_1 = x_k u_2 \Rightarrow x_n = x_k u_2 u_1^{-1}$

contradicting that $x_n \notin \bigcup_{j=1}^{n-1} x_j U^2$.

$\Rightarrow G \supset \bigcup_{j=1}^{\infty} x_j U \Rightarrow m(G) \geq \sum_{j=1}^{\infty} m(x_j U) = \infty$ since $m(x_j U) = m(U) > 0$.

Homogeneous spaces

G top. gp

$H < G$ sbgp. $\Rightarrow G \curvearrowright G/H$ via left transl $(g, g'H) = gg'H$ and $p : G \longrightarrow G/H$ is a G-map, that is it commutes with the actions of G on G and on G/H.

We endow G/H with the quotient topology: $U \subset G/H$ open $\iff p^{-1}(U) \subset G$ is open.

Proposition (1) p is open (p(open) is open)

(2) $G \curvearrowright G/H$ is continuous

(3) G/H is Hausdorff $\iff$ H closed

(4) G is loc. cpt. $\Rightarrow G/H$ loc. cpt.

(5) G loc. cpt. & $C \subset G/H$ is compact $\Rightarrow \exists\ K \subset G$ cpt s.t. $p(K) = C$.

Rk If $G \curvearrowright X$ transitively $\Rightarrow$ X is a G-orbit and it can be identified with G/G_x, where $G_x = \mathrm{Stab}_G(x) = \{g \in G : gx = x\}$.

$$G/G_x \longrightarrow X$$
$$[g] \longmapsto gx$$

Example (1) $G = SO(n+1, \mathbb{R}) = O(n+1, \mathbb{R}) \cap SL(n+1, \mathbb{R})$ acts transitively on $S^n \subset \mathbb{R}^{n+1}$. Let $e_{n+1} \in S^n$. Then

$$SO(n+1, \mathbb{R})_{e_{n+1}} = \left(\begin{array}{c|c} SO(n,\mathbb{R}) & 0 \\ \hline 0 & 1 \end{array}\right) \simeq SO(n, \mathbb{R})$$

$\Rightarrow SO(n+1,\mathbb{R})/SO(n,\mathbb{R}) \cong S^n$.

Example (2) $H^2_{\mathbb{R}} = \{x+iy \in \mathbb{C} : y>0\}$.

$H^2_{\mathbb{R}}$ is a transitive $SL(2,\mathbb{R})$-space, where if $g\in\begin{pmatrix}a&b\\c&d\end{pmatrix}$, $z\in H^2_{\mathbb{R}}$, the action is $\begin{pmatrix}a&b\\c&d\end{pmatrix} z := \frac{az+b}{cz+d}$

(fractional linear transformation).

In fact if $i\in H^2_{\mathbb{R}}$, then

$$\begin{pmatrix} y^{1/2} & xy^{-1/2}\\ 0 & y^{-1/2}\end{pmatrix} i = x+iy.$$

Moreover it is easy to see that $SL(2,\mathbb{R})_i = SO(2,\mathbb{R})$, so that

$$H^2_{\mathbb{R}} \cong SL(2,\mathbb{R})/SO(2,\mathbb{R}).$$

Example (3) $SL(2,\mathbb{R})$ acts also on $\mathbb{R}\cup\{\infty\}$ by f.l.t. with stabilizer G_∞

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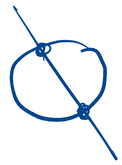
$P = \left\{\begin{pmatrix}a&b\\0&a^{-1}\end{pmatrix} : a,b\in\mathbb{R},\ a\neq 0\right\} = SL(2,\mathbb{R})_\infty$

$SL(2,\mathbb{R}) \curvearrowright H^2_{\mathbb{R}}$

$SL(2,\mathbb{R}) \curvearrowright \mathbb{R}\cup\{\infty\}$

$\mathbb{R}\cup\{\infty\}\ (\cancel{\cong S^1}) \cong \mathbb{P}^1(\mathbb{R}) = \mathbb{P}(\mathbb{R}^2) := \{V\subset \mathbb{R}^2 : \dim V = 1\}$.



Consider $\mathbb{P}^{n-1}(\mathbb{R}) = \mathbb{P}(\mathbb{R}^n) := \{V\subset\mathbb{R}^n : \dim V=1\}$

$\Rightarrow SL(n,\mathbb{R})\curvearrowright \mathbb{P}^{n-1}(\mathbb{R})$ transitively and

$$P = SL(n,\mathbb{R})_{[e_1]} = \left\{\left(\begin{array}{c|c} a & x\\ \hline 0 & A\end{array}\right) : a\in\mathbb{R},\ a\neq0,\ x\in\mathbb{R}^{n-1},\ A\in GL(n-1,\mathbb{R}),\ \det A = a^{-1}\right\}$$

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$\Rightarrow \mathbb{P}^{n-1}(\mathbb{R}) \cong SL(n,\mathbb{R})/P$.

Example (4) $X := \mathrm{Sym}^+_1(n) =$ pos. defin., symm. matrices with det 1. $SL(n,\mathbb{R}) \curvearrowright X$ trans. via $gA := g^t A g$, $g\in SL(n,\mathbb{R})$, $A\in \mathrm{Sym}^+_1(n)$.

Also $SL(n,\mathbb{R})_{Id_n} = \{g\in SL(n,\mathbb{R}) : g^t Id_n g = Id_n\} = SO(n,\mathbb{R}) \Rightarrow$

$\Rightarrow \mathrm{Sym}^+_1(n) \cong SL(n,\mathbb{R})/SO(n,\mathbb{R})$

(Siegel upper half space).

Example (5) $L := \{\mathbb{Z}f_1+\dots+\mathbb{Z}f_n : f_i\in\mathbb{R}^n,\ \det(f_1,\dots,f_n)=1\}$

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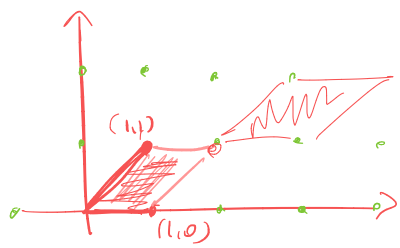

$\det(f_1,f_2) =$ vol

$f_2 = \begin{pmatrix}1\\1\end{pmatrix}$, $f_1 = \begin{pmatrix}1\\0\end{pmatrix}$

The gp $SL(n,\mathbb{R})$ acts on L via $g(\mathbb{Z}f_1+\dots+\mathbb{Z}f_n) := \mathbb{Z}gf_1+\dots+\mathbb{Z}gf_n$

$\det(gf_1,\dots,gf_n) = \det(f_1,\dots,f_n)$ and the stabilizer is $SL(n,\mathbb{Z})$

$\Rightarrow L \cong SL(n,\mathbb{R})/SL(n,\mathbb{Z})$

Back to the proof of b: G l.c. Hausdorff, $\boxed{H\trianglelefteq G}$ closed $\Rightarrow \Delta_G|_H = \Delta_H$

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Pf we define $A_H : C_c(G) \to C_c(G/H)$

$A_H f =: f^H$ is defn. as

$$(A_H f)(\dot{x}) := \int_H f(xh)\, dh,$$

where dh is the left Haar (m) on H. In fact:

(1) $f^H(xh) = f^H(x)$ $\forall x \in G, h \in H$

(2) f u-f. cont. Continuity of int. w.r.t. parameters $\Rightarrow$ $\Rightarrow f^H$ cont.

(3) f cpt. supp $\Rightarrow f^H$ cptly supp.

Claim The functional

$$m(f) := \int_{G/H} f^H(\dot{x})\, d\dot{x}$$

is a left Haar (m) on G, where $d\dot{x}$ is the left Haar (m) on the group G/H.

To verify we need to see that

$$m(\lambda(g) f) = m(f) \quad \forall g \in G,\ f \in C_c(G)$$

$$m(\lambda(g) f) = \int_{G/H} (\lambda(g) f)^H(\dot{x})\, d\dot{x} \underset{\substack{\text{right \& left action} \\ \text{commute}}}{=}$$

$$= \int_{G/H} (\lambda(g) f^H)(\dot{x})\, d\dot{x} \underset{\text{left inv. of } d\dot{x}}{=}$$

$$= \int_{G/H} f^H(\dot{x})\, d\dot{x} = m(f)$$

We saw $m(\rho(g) f) = \Delta_G(g)\, m(f)$

But if $t \in H$

$$m(\rho(t) f) = \int_{G/H} (\rho(t) f)^H(\dot{x})\, d\dot{x} =$$

$$= \int_{G/H} \left(\int_H (\rho(t) f)(xh)\, dh \right) d\dot{x} \underset{\substack{\text{properties} \\ \text{of } dh \\ \text{l.H. (m)}}}{=}$$

$$= \int_{G/H} \left(\int_H \Delta_H(t)\, f(xh)\, dh \right) d\dot{x}$$

$$= \Delta_H(t) \int_{G/H} \left(\int_H f(xh)\, dh \right) d\dot{x}$$

$$= \Delta_H(t) \int_{G/H} f^H(\dot{x})\, d\dot{x}$$

$$= \Delta_H(t)\ m(f).$$

$$m(\rho(t) f) = \boxed{\Delta_H(t)}\ m(f)$$

$$m(\rho(g) f) = \boxed{\Delta_G(g)}\ m(f)$$

Corollary G l.c. Hausdorff, $H \leq G$ closed sbgp. s.t. on G/H $\exists$ a left invariant measure. Then $\Delta_G|_H = \Delta_H$.

Theorem G l.c. Hausdorff, $H \leq G$ closed subgp. Then $\exists$ a left invariant (m) on G/H iff $\Delta_G|_H = \Delta_H$.

Such a (m) $d\dot{x}$ is charact. by Weil formula: $\forall f \in C_c(G)$

$$\int_G f(g)\, dg = \int_{G/H} \left(\int_H f(xh)\, dh \right) d\dot{x}$$

Corollary (1) If H, G are unimod $\Rightarrow \exists!$ (up to scalars) G-inv. m on G/H.

(2) If $H_1 \leq H_2 \leq G$ are all closed and unimodular $\Rightarrow$
$\Rightarrow \exists$ inv. m on G/H_1, G/H_2 and H_2/H_1 and

$$\int_{G/H_1} f(g)\,dg = \int_{G/H_2} \left(\int_{H_2/H_1} f(yz)\,dz \right) dy$$

Pf ($\Rightarrow$) done
($\Leftarrow$) next time. ☐

Next time: examples
Mackey's thm.

Mackey's thm.:

If G, H are l.c.s.c. Hausdorff gps and $\varphi: G \to H$ is a measurable homomorphism $\Rightarrow$
$\Rightarrow \varphi$ is continuous.