07 October 2020

Gleason–Montgomery–Zippin's Th

A top. gp is a Lie gp $\iff$ it has no small subgps (i.e. $\exists\, N \ni e$ s.t. $\nexists$ subgp. contained in N).

Thm G l.c. Hausdorff top-gp. $H < G$ closed sbgp. Then $\exists$ a positive inv. m on G/H $\iff$ $\Delta_G|_H = \Delta_H$.

Ex. $\mathcal{H}^2_{\mathbb{R}} \cong SL(2,\mathbb{R})/SO(2,\mathbb{R})$

$SO(2,\mathbb{R})$ cpt $\Rightarrow$ unimodular

$SL(2,\mathbb{R}) = [SL(2,\mathbb{R}), SL(2,\mathbb{R})]$

For $SL(2,\mathbb{R}) = \left\langle \begin{pmatrix} a & b \\ 0 & a^{-1}\end{pmatrix}, \begin{pmatrix} a & 0 \\ b & a^{-1}\end{pmatrix} \right\rangle$

$$\left[\begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix}, \begin{pmatrix} 1 & x \\ 0 & 1\end{pmatrix}\right] = \begin{pmatrix} 1 & (a^2-1)x \\ 0 & 1\end{pmatrix}$$

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$\Rightarrow \exists$ pos. inv. m on $\mathcal{H}^2_{\mathbb{R}}$

$z \in \mathcal{H}^2_{\mathbb{R}}$, $g = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in SL(2,\mathbb{R})$,

$w = g.z = \frac{az+b}{cz+d}$

- $\operatorname{Im} w^2 = |cz+d|^{-4} \operatorname{Im} z^2$

$dw = (cz+d)^{-2}\, dz$

- $dw\, d\bar{w} = |cz+d|^{-4}\, dz\, d\bar{z}$

$$\Rightarrow \frac{dw\, d\bar{w}}{\operatorname{Im} w^2} = \frac{dz\, d\bar{z}}{\operatorname{Im} z^2} \Rightarrow \text{inv. } m$$

$dz\, d\bar{z} = -2i\, dx\, dy \Rightarrow$ take the abs. value $\boxed{\frac{dx\, dy}{y^2}}$

$$\int_{SL(2,\mathbb{R})} f(g)\, dg = \int_{\mathcal{H}^2_{\mathbb{R}}} \left(\int_{SO(2,\mathbb{R})} f(gk)\, dk \right) \frac{dx\, dy}{y^2}$$

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where $gk \mapsto gi = x+iy$

$P = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1}\end{pmatrix} : a,b \in \mathbb{R},\ a \neq 0 \right\}$

P acts on $\mathcal{H}^2_{\mathbb{R}}$ transitively & almost freely

$P^+ = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1}\end{pmatrix} : a,b \in \mathbb{R},\ a > 0 \right\}$

acts trans. & freely on $\mathcal{H}^2_{\mathbb{R}}$.

$P \cong \mathbb{R} \rtimes \mathbb{R}^*$, $P^+ \cong \mathbb{R} \rtimes \mathbb{R}_{>0}$

in fact every element of P^+ can be written uniquely as a product of an elem. of $\mathbb{R}$ and one of $\mathbb{R}_{>0}$

$$\underbrace{\begin{pmatrix} 1 & x \\ 0 & 1\end{pmatrix}}_{n(x)} \underbrace{\begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2}\end{pmatrix}}_{=a(y)} i = \begin{pmatrix} 1 & x \\ 0 & 1\end{pmatrix} yi = x+iy$$

$N \cong \mathbb{R}$, $A^+ = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix} : a > 0 \right\} \cong \mathbb{R}_{>0}$

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$$\begin{tikzcd} NA^+ \arrow[r] \arrow[rr, bend right, "\Phi"'] & \mathcal{H}^2_{\mathbb{R}} \arrow[r, "\varphi"] & \mathbb{R} \end{tikzcd}$$

$$\int_{\mathcal{H}^2_{\mathbb{R}}} \varphi(x+iy) \frac{dx\, dy}{y^2} = \int_{\mathbb{R}_{>0}} \int_{\mathbb{R}} \Phi(n(x)a(y)) \quad dx \frac{dy}{y^2}$$

$\mathcal{H}^2_{\mathbb{R}} \rightsquigarrow P^+$, $\mathbb{R}_{>0} \rightsquigarrow P^+/N$, $\mathbb{R} \rightsquigarrow N$

$$\Rightarrow \int_{SL(2,\mathbb{R})} f(g)\, dg = \int_{\mathbb{R}_{>0}} \int_{\mathbb{R}} \int_{SO(2)} f(n(x)a(y)k)\, dk\, dx \frac{dy}{y^2}$$

$dk = \frac{d\theta}{2\pi}$, $SO(2) = \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{pmatrix} : \theta \in [0, 2\pi) \right\}$

Gener. Weil formula G, H as before,

$\exists$ a quasi-inv. m on G/H $\iff$ $\exists\, \chi : G \to (\mathbb{R}^*, \cdot)$ homom. s.t. $\Delta_G|_H = \Delta_H\, \chi|_H$

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A measure $m_\lambda^{m\times}$ is quasi-inv. if $\exists$ $\chi: G \longrightarrow (\mathbb{R}^*, \cdot)$ s.t.

$$(g_* m)(A) = m(A)\,\chi(g)$$

Observe also that

$$n(x)\,a(y) = a(y)\,a(y)^{-1}n(x)\,a(y) = a(y)\,n(y^{-1}x)$$

$$\boxed{\int_{SL(2,\mathbb{R})} f(g)\,dg = \int_{\mathbb{R}_{>0}}\int_{\mathbb{R}}\int_{SO(2)} f(a(y)n(x)k)\,dk\,dx\,\frac{dy}{y}}$$

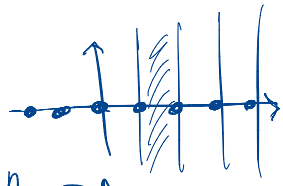
Definition Let G be a l.c. gp. A lattice $\Gamma < G$ is a sbgp s.t.

(1) Γ is discrete

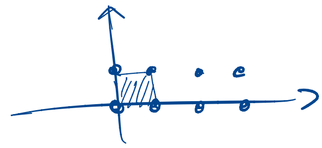
(2) $\exists$ a finite G-inv. m on G/Γ.

Rk G must be unimodular in order to have lattice sbgps.

(1) $\mathbb{Z} < \mathbb{R}^2$ not a lattice



(2) $\mathbb{Z}^2 < \mathbb{R}^2$, $\mathbb{Z}^n < \mathbb{R}^n$ are lattices.

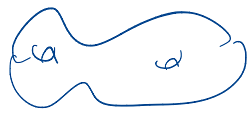


(3) $SL(2,\mathbb{Z}) < SL(2,\mathbb{R})$

In general $G < SL(n,\mathbb{R})$

$G_{\mathbb{Z}} := G \cap SL(n,\mathbb{Z})$ lattice $n \geq 3$.

Σ top. surface $g > 1$



$\pi_1(\Sigma) \xrightarrow{\rho} SL(2,\mathbb{R})$ discrete & injective

$\Rightarrow \Sigma$ hyp surface & $\rho(\pi_1(\Sigma))$ is a lattice in $SL(2,\mathbb{R})$.

Pf $A_H : C_c(G) \longrightarrow C_c(G/H)$

$f \longmapsto f^H =: A_H(f)$,

$$f^H(\dot{x}) = \int_H f(xh)\,dh$$

We need also that A_H is surjective.

Claim If $A_H(f_1) = A_H(f_2)$ and $\Delta_G|_H = \Delta_H$ $\Rightarrow$

$$\Rightarrow \int_G f_1(g)\,dg = \int_G f_2(g)\,dg$$

Define a functional

$m: C_c(G/H) \longrightarrow \mathbb{R}$

$F \longmapsto \int_G f(g)\,dg$,

where $f^H = F$. $\Rightarrow$

$$\int_G f(g)\,dg \overset{\text{defn}}{=} m(F) \overset{\text{RRT}}{=} \int_{G/H} F(\dot{x})\,d\dot{x} =$$

$$= \int_{G/H}\left(\int_H f(xh)\,dh\right)d\dot{x}$$

that is Weil formula.

To prove the claim it is enough to show that if

$$f^H = 0 \Rightarrow \int_G f(g)\,dg = 0.$$

This will follow from

$$\int_G f_1(g)\Big(\underbrace{\int_H f_2(gh)\,dh}_{=0}\Big)dg =$$

$$= \int_G f_2(g)\left(\int_H f_1(gh)\,dh\right)dg$$

In fact if $f_2^H = 0 \Rightarrow$

$$\Rightarrow 0 = \int_G f_2(g)\left(\int_H f_1(gh)\,dh\right)dg$$

It would be hence enough to find $f_1 \in C_c(G)$ s.t. $f_1^H \equiv 1$. (**) But this exists because of surjectivity. (To be checked)

To prove (*) use Fubini

$$\int_G f_1(g)\left(\int_H f_2(gh)\,dh\right)dg =$$

$$= \int_H \int_G f_1(g) f_2(gh)\,dg\,dh$$

$$= \int_H \left(\int_G f_1(gh^{-1}) f_2(g) \Delta_G(h)^{-1} dg\right) dh$$

$$= \int_G \left(\int_H f_1(gh^{-1}) \Delta_G(h)^{-1} dh\right) f_2(g)\,dg$$

$$= \int_G \left(\int_H f_1(gh) \underbrace{\Delta_H(h^{-1}) \Delta_G(h)}_{=1} dh\right) f_2(g)\,dg$$

$$= \int_G \int_H f_1(gh) f_2(g)\,dh\,dg \quad \square$$

Thm (Mackey) G, H l.c. Haud. 2nd countable groups, $\varphi: G \to H$ meas. homo. Then φ is continuous.

(**) We only need to find $f_1 \in C_c(G)$ such that $f_1^H \equiv 1$ on supp f_2 - Since supp f_2 is compact, life is good!