Lie groups exercise class #2

Topics:

1. Operator topologies (beyond separable $\mathcal{H}$)

2. Ex. 3 - solution

did not have time → (3. Two ways to represent $\mathbb{Z}_p$)

4. Orbit-stabilizer: topological version

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1 Operator topologies

- Hilbert space $\mathcal{H}$. We have ~~defined~~ defined the weak / strong operator topologies on $\mathcal{U}(\mathcal{H})$ as:

$$T_n \xrightarrow{SOT} T \text{ iff } T_n x \longrightarrow Tx \in \mathcal{H} \quad \forall x.$$

$$T_n \xrightarrow{WOT} T \text{ iff } \lambda(T_n x) \to \lambda(Tx) \in \mathbb{R}/\mathbb{C}$$
$$\lambda \in \mathcal{H}^*, x \in \mathcal{H}$$

Q: Are we allowed to do this?

→ Yes, if $\mathcal{U}(\mathcal{H})$ is first-countable.

Def: ① Strong operator topology:

$$\underbrace{\mathcal{U}_s(T_0, x, \varepsilon)}_{T_0 \in \mathcal{U}(\mathcal{H}),\ x \in \mathcal{H},\ \varepsilon > 0} := \{ T \in \mathcal{U}(\mathcal{H}) : \|T x - T_0 x\| < \varepsilon \}$$

nbh sub basis at T_0.

Exercise: This defines a topology, and convergence is as previously defined.

② Weak operator topology:

$$\underbrace{\mathcal{U}_w(T_0, \lambda, x, \varepsilon)}_{T_0 \in \mathcal{U}(\mathcal{H}),\ \lambda \in \mathcal{H}^*,\ \dots} = \{ T \in \mathcal{U}(\mathcal{H}) : |\lambda(Tx - T_0 x)| < \varepsilon \}$$

Exercise: See above.

Lemma: If $\mathcal{H}$ is separable.

then WOT + SOT are first-countable.

LP4: SOT: $T_0 \in \mathcal{U}(\mathcal{H})$. Nbh $\overset{\text{sub}}{\text{basis}}$

is $\{\mathcal{U}_s(T_0, x, \varepsilon) : x \in \mathcal{H}, \varepsilon > 0\}$.

$(x_n)_{n \geq 1}$ countable dense set:

$$A_n = \bigcap_{i=1}^{n} \mathcal{U}_s(T_0, x_i, \tfrac{1}{i})$$ is a

countable nbh basis of T_0.

WOT: $\{\mathcal{U}_w(T_0, \lambda, x, \varepsilon) : \lambda \in \mathcal{H}^*,$
$x \in \mathcal{H}, \varepsilon > 0\}$

$\mathcal{H}^*$ is separable $\Rightarrow \exists\, (\lambda_n)_{n \geq 1}$
countable dense

$$A_n = \bigcap_{i,j=1}^{n} \mathcal{U}_w(T_0, \lambda_i, x_j, \tfrac{1}{n})$$

$\square$

- This shows that you can ~~an~~ use Convergence in EX 1, since in class we assumed $\mathcal{H}$ to be separable.

You can also use these definitions, and then don't even need separability!

EX 3 - Solution

Two topologies on $GL_n(\mathbb{R})$:

$\mathcal{T}_E$: Given by $GL_n(\mathbb{R}) \subseteq \mathbb{R}^{n \times n}$

distance $\|A - B\|_{n\times n} = \sqrt{\sum |a_{ij} - b_{ij}|^2}$

$\mathcal{T}_{co}$: Given by $GL_n(\mathbb{R}) \subseteq C(\mathbb{R}^n, \mathbb{R}^n)$

subbasis $S(\underset{\substack{\uparrow \\ \text{compact}}}{K}, \underset{\substack{\uparrow \\ \text{open}}}{U}) = \{ A \in GL_n(\mathbb{R}) : A(K) \subseteq U \}$

<u>Goal</u>: $\mathcal{T}_E = \mathcal{T}_{co}$.

$*$: $\mathcal{T}_{co} \subseteq \mathcal{T}_E$. $A \in GL_n(\mathbb{R})$

$A \in S(K, U)$. To show:

$\exists\ \varepsilon > 0 : \quad A \in B_\varepsilon(A) \subseteq S(K, U)$.

$\underbrace{A(K)}_{\text{compact}} \subseteq \underbrace{U}_{\text{open}} \implies$

$\implies \exists\ \delta > 0$ s.t. $B_\delta(A(K)) \subseteq U$.

Want: $\varepsilon > 0$ s.t. $B \in B^{n\times n}_\varepsilon(A) \implies B(K) \subseteq B^n_\delta(A(K))$,

$K \ni x = \sum_{i=1}^{n} \lambda_i e_i$. ~~...~~ λ upper bound over all x.

$$\|A(x) - B(x)\|_n = \|\sum_{i=1}^{n} \lambda_i (\underset{\text{columns}}{a_i - b_i})\|_n \leq$$

$$\leq |\lambda| \sum_{i=1}^{n} \underbrace{\|a_i - b_i\|_n}_{\leq \|A-B\|_{n\times n}} \leq |\lambda| \cdot n \cdot \|A-B\|_{n\times n}.$$

So if $\varepsilon = \dfrac{\delta}{|\lambda| \cdot n}$, done.

$\supseteq$: $\mathcal{T}_{\mathrm{co}\varepsilon} \subseteq \mathcal{T}_{\mathrm{co}}$. $A \in GL_n(\mathbb{R})$

$A \in B_\varepsilon(A)$. To show:

$\exists$ K_i compact, U_i open $(i \leq K < \infty)$:

$$A \in \bigcap_{i=1}^{K} S(K_i, U_i) \subseteq B_\varepsilon(A)$$

$K_i = \{e_i\}$, $i = 1, \dots, n$.

$U_i = B_\delta(a_i)$ for some $\delta > 0$.

$$B \in \bigcap_{i=1}^{n} S(\{e_i\}, B_\delta(e_i)).$$

$$\|A-B\|_{n\times n} \leq \sum_{i=1}^{n} \|a_i - b_i\|_n =$$

$$= \sum_{i=1}^{n} \|A\cdot e_i - B\cdot e_i\|_n < n\cdot\delta$$

$\Rightarrow$ if $\delta = \frac{\varepsilon}{n}$, done

/ Ex. 3

Orbit-stabilizer for topological groups

- Recall from group theory that if $G \curvearrowright X$ transitively $(G\cdot x = X \ni x)$, then $G/G_x \longrightarrow X : gG_x \longmapsto g\cdot x$ is bijective.

(l.c.H. = locally compact Hausdorff)

Prop°: $G \curvearrowright X$. G l.c.H. + separable, X l.c.H., $G\times X \longrightarrow X$, $(g,x) \longmapsto g\cdot x$ continuous.

Then $G/G_x \longrightarrow X : gG_x \longmapsto g\cdot x$ is a homeomorphism.

Rmk: G is a Lie group, X a manifold, then hypotheses are satisfied.

Pf: $\varphi : G \to X : g \mapsto g \cdot x_0$, $x_0 \in X$.

$\bar{\varphi} : G/G_{x_0} \to X$ is bijective.

φ continuous $\Rightarrow$ $\bar{\varphi}$ continuous (def of quotient topology)

φ open $\Rightarrow$ $\bar{\varphi}$ open.

$\hookrightarrow$ $\bar{\varphi}(U) = \varphi(\overbrace{\pi^{-1}(U)}^{\text{open}})$ open

$\hookrightarrow$ To show: $\varphi(U)$ open $\forall$ $U \subseteq G$ open.

Enough to show: for U open, precompact.

b/c $\longrightarrow$ form a basis of G l.c.H.

Enough to show: for $e \in U$. Indeed,

$g \in V$ precompact. $\Rightarrow$ $g^{-1}V \ni e$ $\Rightarrow$

$\overset{\text{hyp}}{\Longrightarrow}$ $\varphi(g^{-1}V)$ open $\underset{G\text{-map}}{\overset{\varphi}{\Longrightarrow}}$ $g^{-1}\varphi(V)$ open

$\underset{\text{by homeo}}{\overset{G \curvearrowright X}{\Longrightarrow}}$ $\varphi(V)$ open.

Enough to show: $\varphi(e) \in \operatorname{int}(\varphi(U))$.

$\varphi(g^{\in U}) \in \varphi(U)$. Then

$$\varphi(e) = g^{-1} \cdot \varphi(g) \in g^{-1} \cdot \varphi(U)$$

$$\in \varphi(\underbrace{g^{-1} U}_{\ni e}) . \overset{hyp}{\Longrightarrow}$$

$$\Rightarrow \varphi(e) \in \operatorname{int}(\varphi(g^{-1} U))$$

$$\Rightarrow \varphi(g) \in g \cdot \operatorname{int}(\varphi(g^{-1}(U))) =$$

$$= \operatorname{int}(\varphi(U))$$

Enough to show: $\operatorname{int}(\varphi(U)) \neq \emptyset$.

Let $e \in W \subseteq U$, W open, precompact

s.t. $W^2 \subseteq U$, $W^{-1} = W$.

$$\overset{hyp.}{\Longrightarrow} \operatorname{int}(\varphi(W)) \neq \emptyset \Rightarrow$$

$$\Rightarrow \exists\, h \in W : \varphi(h) \in \overset{\text{open}}{O} \subseteq \varphi(W)$$

$$\Rightarrow \varphi(e) = h^{-1} \cdot \varphi(h) \in h^{-1} \cdot \overset{\text{open}}{O} \subseteq$$

$$\subseteq \varphi(\underbrace{h^{-1} W}_{\subseteq W^2}) \subseteq \varphi(U)$$

$\Rightarrow \varphi(e) \in \operatorname{int}(\varphi(U))$.

Enough to show: $\operatorname{int}(\varphi(\overline{U})) \neq \emptyset$.

Need to use:

Lemma: l.c.H. spaces are regular:

$\forall\, x \in U^{\leftarrow \text{open}} \quad \exists \quad V$ open s.t.

$$x \in V \subseteq \overline{V} \subseteq U.$$

"Pf": Move to the one-point compactification.

G is regular $\Rightarrow \exists \quad e \in V \subseteq \overline{V} \subseteq U$
$\uparrow$ precompact

$$\overset{\text{hyp.}}{\Longrightarrow} \quad \underset{\emptyset}{\underset{\#}{\operatorname{int}(\varphi(U))}} \supseteq \operatorname{int}(\varphi(\overline{V})) \neq \emptyset$$

✓

Proof of $\operatorname{int}(\varphi(\overline{U})) \neq \emptyset$

G is separable: $\exists\ (d_n)_{n \geq 1}$ dense.

$$X = \varphi(G) \overset{\text{density}}{=} \varphi\Big(\bigcup_{n \geq 1} d_n \cdot U\Big) =$$

$$= \bigcup_{n\geq 1} d_n \cdot \varphi(U) \subseteq \bigcup_{n\geq 1} \underbrace{d_n \cdot \varphi(\underbrace{\overline{U}}_{\text{compact}})}_{\text{compact}}$$

X is Hausdorff $\Longrightarrow$ $d_n \cdot \varphi(\overline{U})$ is closed $\forall n$.

THM (Baire category): If X is l.c.H.,

$X = \bigcup_{n\geq 1} \underset{\text{closed}}{F_n} \Longrightarrow \exists\, n \geq 1$

s.t. $\operatorname{int}(F_n) \neq \emptyset$.

$\Longrightarrow \exists\, n \geq 1 : \operatorname{int}(d_n\, \varphi(\overline{U})) \neq \emptyset$

$\Longleftrightarrow \operatorname{int}(\varphi(\overline{U})) \neq \emptyset$. $\blacksquare$