14 October 2020

Thm (Mackey) G, H l.c.s.c. gps. $\varphi : G \to H$ measurable $\Rightarrow \varphi$ is continuous.

Pf Assume φ is onto.
Want to show that $\forall\, V \ni e_H$ open nbd. $\exists$ open nbd $N \ni e_G$ s.t. $\varphi(N) \subset V$ or $N \subset \varphi^{-1}(V)$.
Let $U \subset V$ an open symm. nbd. of e_H s.t. $U^2 \subset V$ and let $(h_n)_{n \in \mathbb{Z}}$ be a countable dense set $\Rightarrow$

$$\Rightarrow H = \bigcup_{n \in \mathbb{N}} h_n U. \text{ Let } (g_n)_{n \in \mathbb{N}} \subset G$$

s.t. $\varphi(g_n) = h_n \Rightarrow G = \bigcup_{n \in \mathbb{N}} g_n \varphi^{-1}(U)$.

If m is the (left) Haar (m)

$$\Rightarrow m(G) > 0 \Rightarrow 0 < \underset{\text{for some } n}{m}\left(g_n \varphi^{-1}(U)\right) =$$

$$= m(\varphi^{-1}(U)) \Rightarrow m(\varphi^{-1}(U)) > 0.$$

Inner regularity $\Rightarrow \exists\, A \subset G$ cpt s.t. $A \subset \varphi^{-1}(U)$ and

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$m(A) > 0$. It is enough to find an open nbd N s.t.

$$\varphi^{-1}(V) \supset \varphi^{-1}(U)\varphi^{-1}(U) \supset \boxed{AA^{-1} \supset N}$$

Lemma If $A \subset G$ is compact with $m(A) > 0 \Rightarrow \exists$ open nbd $N \ni e_G$ s.t. $N \subset AA^{-1}$.

Pf If $Ax \cap A \neq \emptyset \Rightarrow x \in AA^{-1}$
$\Rightarrow AA^{-1} \supset \{x : Ax \cap A \neq \emptyset\}$ hence it is enough to find an open nbd in $\{x : Ax \cap A \neq \emptyset\} \supset N \ni e_G$.
Outer regular $\Rightarrow \exists\, W \supset A$ open set s.t. $2m(A) > m(W)$

Claim Enough to find $N \ni e_G$ s.t. $AN \subset W$. If so then $\forall x \in N$

$$\tfrac{1}{2} m(W) < m(A) = m(Ax) < m(W).$$

If we have proven the claim we are done. In fact if $x \in N$ and $Ax \cap A = \emptyset$
$\Rightarrow m(Ax \cup A) = m(Ax) + m(A)$
$\Rightarrow 2\, m(A) > m(W)$ ↯

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since $m(W) > m(Ax \cup A)$.
So we need to prove the claim.
For $x \in A$ let $V_x \ni e_G$ be s.t. $x V_x \subset W$. Let U_x be a symm. open nbd of $e_G \in G$ with $U_x U_x \subset V_x$
$\Rightarrow \{x U_x : x \in A\}$ is an open cover of $A \Rightarrow A \subset x_1 U_{x_1} \cup \dots \cup x_n U_{x_n}$

$N := U_{x_1} \cap \dots \cap U_{x_n} \subset U_{x_j}\ \forall j$

$$\begin{aligned} \Rightarrow AN &= x_1 U_{x_1} N \cup \dots \cup x_n U_{x_n} N \\ &= x_1 U_{x_1}^2 \cup \dots \cup x_n U_{x_n}^2 \\ &\subseteq x_1 V_{x_1} \cup \dots \cup x_n V_{x_n} \subset \\ &\subset W. \end{aligned}$$

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Defn. A Lie gp is a top. gp. with the structure of smooth manifold & such that the group operations are smooth. The dim of the Lie gp is the dim. of the underlying mfd.

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Ex. $(\mathbb{R}^n, +)$, $GL(n, \mathbb{R}) \subset \mathbb{R}^{n^2}$,

Ex. countable discrete gps are Lie gps, and are 0-dim. **Ex.** Lattice, $SL(2,\mathbb{Z}) < SL(2,\mathbb{R})$, $\mathbb{Z}^n < \mathbb{R}^n$

Ex. Inverse limit of discrete gps is not a Lie gp. Ex. profinite gps.

Ex. X top. space $\Rightarrow$ Homeo(X) is often not l.c. $\Rightarrow$ not Lie gp.

Ex. (X,d) cpt metric space $\Rightarrow$
$\Rightarrow$ Iso(X,d) might be a Lie gp as it is l.c. For ex. $\mathrm{Iso}(\mathbb{R}^n, d_{\mathrm{Eucl}}) = O(n,\mathbb{R}) \ltimes \mathbb{R}^n$ is a Lie gp.

Ex. $A_{\det} = \left\{ \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} : \lambda_j \in \mathbb{R}^* \right\}$

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is a Lie gp since $A_{ut} \cong (\mathbb{R}^*, \cdot)^n$.

$N = \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{pmatrix} \right\} \cong \mathbb{R}^{\frac{n(n+1)}{2}}$

$\uparrow$ mfd but not as a gp unless $n=2$

Ex: $V \subset \mathbb{R}^n$ k-dim. subspace

$\mathrm{Stab}_{GL(n,\mathbb{R})}(V) = \{ g \in GL(n,\mathbb{R}) : gV \subset V \}$

$= \left\{ g = \left(\begin{array}{c|c} A & B \\ \hline 0 & C \end{array}\right) : \begin{array}{l} A \in GL(k,\mathbb{R}) \\ C \in GL(n-k,\mathbb{R}) \\ B \in M_{k\times(n-k)}(\mathbb{R}) \end{array} \right\}$

$\cong GL(k,\mathbb{R}) \times GL(n-k,\mathbb{R}) \times \mathbb{R}^{k(n-k)}$

$\uparrow$ mfd

Thm G Lie gp, $H < G$ sbgp.
If H is a regular subm.
then H is a Lie gp.

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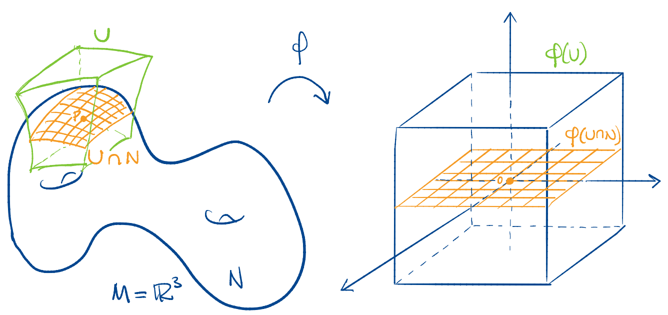
Defn. M smooth n-dim. mfd.

(1) A subset $N \subset M$ has the reg. submanifold property if every $p \in N$ has a nbd (U, φ) with local coord. $(x_1, \dots, x_m)$ s.t.

(a) $\varphi(p) = 0$

(b) $\varphi(U) = (-\varepsilon, \varepsilon)^n$

(c) $\varphi(U \cap N) = \{ x \in (-\varepsilon,\varepsilon)^n : x_{m+1} = \dots = x_n = 0 \}$



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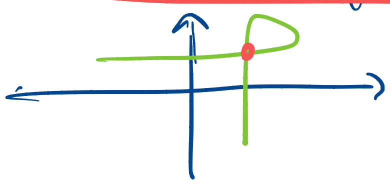
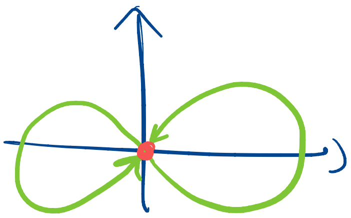
(2) A regular submanifold $N \subset M$ is a subset with the regular subm. property and the smooth structure induced by the nbds defined by the r.s.p.

Ex





Lemma M, M' smooth mfds, $f: M' \to M$ smooth map, $N \subset M$ regular subm. Then $f: M' \to N$ is also smooth.

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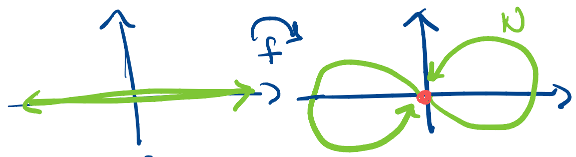
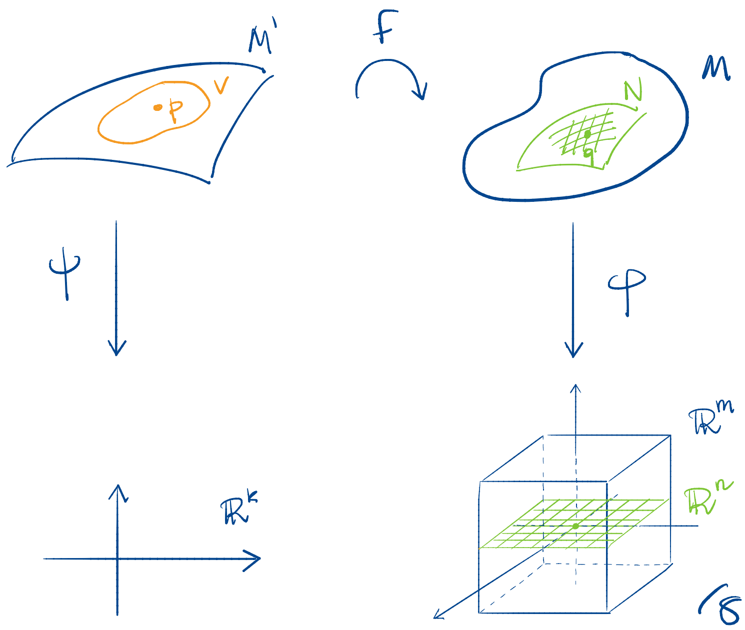
Ex



$f: \mathbb{R} \to \mathbb{R}^2$ is smooth

$f: \mathbb{R} \to N$ not even cont.

Pf $p \in M'$, $f(p) =: q \in M$, (U, φ) a coord. nbd. around q.



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$\varphi(q)=0 \qquad \varphi(U)=(-\varepsilon,\varepsilon)^m$

$\varphi(U\cap N)=\{x\in(-\varepsilon,\varepsilon)^m : x_{n+1}=\dots=x_m=0\}$

(V,ψ) coord. nbd around $p\in M'$ s.t. $\boxed{f(V)\subset U}$ and $x_1,\dots,x_k$ are local coord. in (V,ψ) for M'.

$\Rightarrow$ in local coord. $f: M'\to M$

$$\varphi\circ f\circ\psi^{-1}(\underbrace{x_1,\dots,x_k}_{x})=(f_1(x),\dots,f_n(x),0,\dots,0)$$

But in local coord. $f: M'\to N$ is

$$\varphi\circ f\circ\psi^{-1}(x_1,\dots,x_k)=(f_1(x),\dots,f_n(x))$$

that is the same as $f: M'\to M$ but followed by the proj. $\pi:\mathbb{R}^m\to\mathbb{R}^n$. ∎

Pf of thm. $m: G\times G\to G$ smooth.

$\Rightarrow m: H\times H\to G$ smooth

But this takes values in H and H is a r.s. $\Rightarrow m: H\times H\to H$ is smooth.

Likewise for $i: H\to H$. ∎

How to see whether a subsp. is a submanifold?

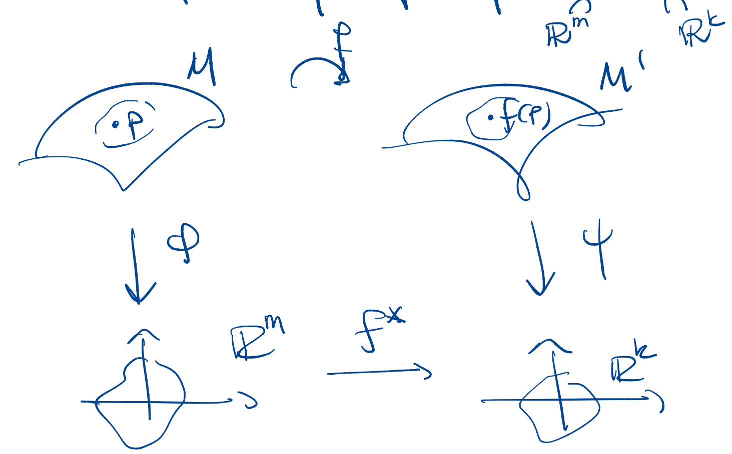
Thm (Inverse Function Thm)

M, M' smooth mfds of dim m, k respectively. $f: M\to M'$ smooth map s.t. rk f is constant. Then $\forall q\in M'$, $f^{-1}(q)$ is a regular subm. of dimension $m-\operatorname{rk} f$.

Recall (1) $p\in M$, $f(p)\in M'$

$(U,\varphi)\ni p$, $(V,\psi)\ni f(p)$

The rk of f at p is the rank of the Jacobian of $f^* := \psi\circ f\circ\varphi^{-1}: U\to V$, $U\subset\mathbb{R}^m$, $V\subset\mathbb{R}^k$



(2) If $f: M\to M'$ is a diffeo $\Rightarrow \operatorname{rk} f=\dim M=\dim M'$

Pf of thm. $m: G\times G\to G$ smooth.

$\Rightarrow m: H\times H\to G$ smooth

But this takes values in H and H is a r.s. $\Rightarrow m: H\times H\to H$ is smooth.

Likewise for $i: H\to H$. ∎

How to see whether a subsp. is a submanifold?

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M, M' smooth mfds of dim m, k respectively. $f: M\to M'$ smooth map s.t. rk f is constant. Then $\forall q\in M'$, $f^{-1}(q)$ is a regular subm. of dimension $m-\operatorname{rk} f$.

Ex. $SL(n,\mathbb{R}) = \{ g \in GL(n,\mathbb{R}) : \det g = 1 \}$

$\Rightarrow \det : GL(n,\mathbb{R}) \longrightarrow \mathbb{R}^*$ is smooth and $\det^{-1}(1) = SL(n,\mathbb{R})$

To prove that $SL(n,\mathbb{R})$ is a Lie gp enough to show $rk\, \det$ is constant.

$$\begin{tikzcd} GL(n,\mathbb{R}) \arrow[r, "\det"] \arrow[d, "L_X"'] & \mathbb{R}^* \arrow[d, "\ell_X"] \\ GL(n,\mathbb{R}) \arrow[r, "\det"] & \mathbb{R}^* \end{tikzcd}$$

$\Rightarrow \det = \ell_X \circ \det \circ L_X$

diffeo

and $rk_A \det = rk_{AX^{-1}} \det$

LHS does not depend on X

$\Rightarrow$ RHS does not depend on X. Take $X = A$ $\Rightarrow$

$\Rightarrow rk_A \det = rk_I \det$. constant.

$\Rightarrow SL(n,\mathbb{R})$ is a Lie gp.

Ex. $rk\, \det = 1$.

$\Rightarrow \dim SL(n,\mathbb{R}) = n^2 - 1$.

Ex. Show that $O(n,\mathbb{R})$ is a Lie gp and compute $\dim O(n,\mathbb{R})$.

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