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Thue (Hackett) $G$, $H$. i.c.c.c. gps.

$q: G 	o H$ measurable $\Rightarrow \phi$ is continuous.

**Proof.** Assume $q$ is onto.

Want to show that if $v \in E_H$ open nbhd. $\exists$ open nbhd $N \ni x$ s.t. $q(N) \subseteq V$ or $N \ni \phi(v)$.

Let $U \subseteq V$ an open symmetric nbhd.

So $E_H$ s.t. $U \ni V$ and let $(h_n)_{n \in \mathbb{N}}$ be a countable dense set $\Rightarrow$

$H = \bigcup h_n U$. Let $(g_n)_{n \in \mathbb{N}}$ s.t. $\phi(g_n) = h_n \Rightarrow G = \bigcup g_n \phi(v)$.

If $m$ is the (left) Haar @

$m(G) > 0 \Rightarrow \exists m \left(g_n \phi(v)\right) = 0$ for some $n$.

$m(\phi(v)) \Rightarrow m(\phi(v)) > 0$.

Inner regularity $\Rightarrow \exists A \subseteq G$ opt s.t. $A \subseteq \phi(v)$ and

$m(A) > 0$. It is enough to find an open nbhd $N$ s.t.

$
\phi(v) \subseteq \phi(u) \phi(u) \subseteq A \cap N
$

**Lemma** If $A \cap G$ is compact with

$m(A) > 0 \Rightarrow \exists$ open nbhd $N \ni x \subseteq A \cap N$.

$\Rightarrow A \cap N = \{x: A \cap N \ni x\}$ hence

it is enough to find an open nbhd in $\{x: A \cap N \ni x\} \ni x$.

Outer regular $\Rightarrow \exists W \supseteq A$ open set s.t. $2 m(A) > m(W)$

**Claim**. Enough to find $N \ni x$ s.t. $AN \cap W$, it so then $x \in W$.

$m(W) < m(A) = m(A) - m(W)$.

Let we have proven the claim we are done. In fact

if $x \in N$ and $A \cap N = \emptyset$

$\Rightarrow m(A \cap N) = m(A) = m(A) \Rightarrow 2 m(A) > m(W)$.

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**Definitions**. A Lie gp is a top. gp. with the structure of smooth manifold $\mathcal{M}$ such that the group operations are smooth.

The dim of the Lie gp is the dim of the underlying $\mathcal{M}$.

Ex. $(\mathbb{R}^1, +)$, $GL(n, \mathbb{R}) \subseteq \mathbb{R}^n$.

11. Ex. Countable discrete gps are Lie gps. and are 0-dim. Ex. Lattice, $SL(2, \mathbb{Z}) < GL(2, \mathbb{R})$, $\mathbb{Z}^2 < \mathbb{R}^2$.

Ex. Inverse limit of discrete gps is not a Lie gp. Ex. Profinite gps.

Ex. $X$ top. space $\Rightarrow$ Homeo $(X)$ is often not l.c. $\Rightarrow$ not Lie gp.

Ex. $(X, d)$ opt metric space $\Rightarrow$

$\Rightarrow$ Iso $(X, d)$ might be a Lie gp as it is l.c. For ex

$\text{Iso}(\mathbb{R}^1, d_{\text{Eucl}}) = \mathbb{R}(n, 1) \times \mathbb{R}^n$ is a Lie gp.

Ex. $A_{d_{\mathbb{R}^2}} = \{X^2: x \in \mathbb{R}^2\}$
is a Lie gp since $A_{n+1} \cong \mathbb{R}^{n+1}_+$. 
$N = \{ (\cdot, \cdot) \} \cong \mathbb{R}^n_{++}$ mfd but not as a gp unless $n=2$.

**Ex:** $N \subset \mathbb{R}^n \$: k-dim. subspae.

$$\text{Stab}_{\text{GL}(n, \mathbb{R})} (V) = \{ g \in \text{GL}(n, \mathbb{R}) : gV = V \}$$

$$= \{ g = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \in \text{GL}(n, \mathbb{R}) : C \in \text{GL}(k, \mathbb{R}), B \in M_{k \times (n-k)}(\mathbb{R}), A \in M_{n \times n}(\mathbb{R}) \}$$

$N \subset \text{GL}(k, \mathbb{R}) \times \text{GL}(n-k, \mathbb{R}) \times \mathbb{R}^{k(n-k)}$.

**Thm:** $G$ Lie gp, $H \subset G$ Lie gp.

If $H$ is a regular subm.
Then $H$ is a Lie gp.

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**Defn.** $M$ smooth $n$-dim. mfd.

(i) A subset $NCM$ is the reg.

submanifold property if every pen has a nbhd

$(U, \varphi)$ with local cond.

$(x_1, \ldots, x_m) = t$.

(a) $\varphi(t) = 0$

(b) $\varphi(U) = (-\epsilon, \epsilon)^n$

(c) $\varphi^{-1}(U \cap N) = \{ x \in (-\epsilon, \epsilon)^n : x_{m+1} = \ldots = x_m = 0 \}$

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**Ex**

$f: \mathbb{R} \to \mathbb{R}^2$ is smooth.

$f: \mathbb{R} \to N$ not even cont.

pf: $p \in M'$, $f(p) = q \in M$, $(U, \varphi)$ a coord. nbhd. around $q$.

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**Lemma.** $M, M'$ smooth mfds.

$f: M' \to M$ smooth map, $NCM$ regular subm. Then

$f: M' \to N$ is also smooth.
ϕ(q) = 0  \quad ϕ(U) = (-ε, ε)^{m} \\
ϕ(V Ω N) = \frac{1}{2} xe(-ε, ε)^{m}: x_{m+1} = ... = x_{n} = 0 \\
(V, φ) \text{ coord. nd } aust \ φ_e M \text{ s.t. } f(\nu) \subset U \text{ and } x_{1},...,x_{k} \text{ are local coord. in } (V, φ) \text{ for } M'.

⇒ \text{ in local coord. } f: M' → M \\
\phi \circ f' (x_{1},...,x_{k}) = (f(x),...f(x),0,0) \\
\text{But in local coord. } f: M' → N_{1} \\
\phi \circ f' (x_{1},...,x_{k}) = (f(x),...f(x)) \\
\text{that is the same as } f: M' → M \text{ but followed by the proj. } π: R^{n} → R^{n}.

Recall (1) \quad \phi \in M, \quad f(\phi) \in M' \\
(U, φ) \supseteq φ, \quad (V, φ) \supseteq f(φ) \\
\text{Then the rank of the Jacobian } 0 \text{ at } φ \text{ is the rank of the Jacobian } 0 \text{ at } f^* = ψ \circ f \circ φ^*: U \supseteq V' \supseteq R^{m} → R^{n}.

(2) \quad \text{If } f: M → M' \text{ is a diffeo } \Rightarrow \text{ rank } f = \text{ dim } M = \text{ dim } M'.

\textbf{Proof of thm.} \quad m: GxG → G smooth. \\
⇒ m: t \times H → G smooth \\
\text{But this takes value in } H \\
\text{and } t \in \omega \text{ s.t. } m: t \times H → H \text{ is smooth.} \\
\text{Likewise for } i: H → H.

\textbf{How to see whether a subsp. is a submanifold?} \\
\textbf{Immerse Function Thm} \\
M, M' \text{ smooth mfd s.t. } dim m, k \text{ respectively. } f: M → M' \text{ smooth map s.t. } \nu \circ f \text{ is constant. Then } f \circ q \in M', \\
f^{-1}(q) \text{ is a regular subm. of dimension } m - \text{ rank } f.$
Ex. \( \text{SL}(n, \mathbb{R}) = \{ g \in \text{GL}(n, \mathbb{R}) : \det g = 1 \} \)

\[ \Rightarrow \det : \text{GL}(n, \mathbb{R}) \rightarrow \mathbb{R}^* \]

is smooth and \( \det(g) = \text{det}(g) \) for all \( g \in \text{GL}(n, \mathbb{R}) \).

To prove that \( \text{SL}(n, \mathbb{R}) \) is a Lie group, we need to show that the determinant is constant.

\[ \text{GL}(n, \mathbb{R}) \xrightarrow{\det} \mathbb{R}^* \]

\[ \text{L}_X \bigg|_{\text{det}} \subseteq \text{det} \]

\[ \text{GL}(n, \mathbb{R}) \xrightarrow{\det} \mathbb{R}^* \]

\[ \Rightarrow \det = b_x \circ \det \circ L_x \]

\[ \Rightarrow \det \text{ is constant} \]

and \( \text{uk}_G \det = \text{uk}_A \det \)

LHS does not depend on \( G \)

\[ \Rightarrow \text{RHS does not depend on } X \text{, take } X = A \Rightarrow \]

\[ \Rightarrow \text{uk}_A \det = \text{uk}_A \det \text{, constant} \]

\( \Rightarrow \text{SL}(n, \mathbb{R}) \) is a Lie group.

Ex. \( \text{uk}_A \det = 1 \),

\[ \Rightarrow \dim \text{SL}(n, \mathbb{R}) = n^2 - 1 \]

Ex. Show that \( Q(n, \mathbb{R}) \)

is a Lie group and compute \( \dim Q(n, \mathbb{R}) \).