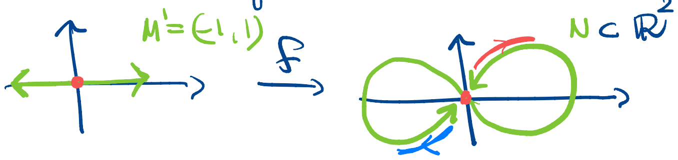
15 October 2020

- In Mackey's thm we need to add Hausdorff.
- $f: G \to H$ smooth between Lie groups $\Rightarrow$ $\mathrm{rk}\, f$ is constant

Yesterday

Lemma If $f: M' \to M$ is a smooth map taking values into $N \subset M$ and N is a regular subm. $\Rightarrow$ $f: M' \to N$ is smooth.

Remark Not true if N is not a regular manifold.



$f(0) = 0$

$f(0,1) =$  $f(t\to 1)$, $f(t \to 0^+)$

$f(-1,0) =$ $f(t\to 0^-)$, $f(t \to -1)$

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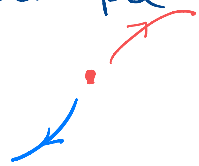
Give N the topology for which it is diffeo to $\mathbb{R}$.

$\Rightarrow$ $f: (-1,1) \to \mathbb{R}^2$ smooth

$f: (-1,1) \to N$ not cont.

because for example

$f(-1,1) =$ 

Aside: $\varphi: N \to M$ constant rk, 1–1

$f: M' \to M$ s.t. $f: M' \to N$ is cont $\Rightarrow$ $f: M' \to N$ smooth.

smooth

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Ex $O(n,\mathbb{R})$ is a Lie group

$= \{X \in GL(n,\mathbb{R}) : {}^tX X = Id\}$.

$f: GL(n,\mathbb{R}) \to GL(n,\mathbb{R})$

$f(X) := {}^tX X$.

$O(n,\mathbb{R}) = f^{-1}(I)$, so we need to show that $\mathrm{rk}\, f$ is constant.

$$f(XA^{-1}) = {}^t(XA^{-1})\, XA^{-1} =$$
$$= {}^tA^{-1}\, {}^tX X A^{-1} =$$
$$= L_{({}^tA^{-1})} \circ R_{A^{-1}} \circ f$$
$$\Rightarrow f = L_{({}^tA^{-1})} \circ R_{A^{-1}} \circ f \circ R_A$$

diffeos

$\mathrm{rk}_X f = \mathrm{rk}_{XA} f \rightsquigarrow \mathrm{rk}_X f = \mathrm{rk}_I f$

One can show that

$\mathrm{rk}\, f = \dfrac{n(n+1)}{2}$ $\Rightarrow$

$\Rightarrow \dim O(n,\mathbb{R}) = n^2 - \dfrac{n(n+1)}{2} = \dfrac{n(n-1)}{2}$.

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M smooth manifold

$C^\infty(M)$ = smooth $\mathbb{R}$-valued functions on M.

$U \subset M$ open set, $C^\infty(U)$

Given $f \in C^\infty(U)$, can extend it to $f^* \in C^\infty(M)$: take $p \in U$, $V \subset U$ nbd of p $\Rightarrow \exists f^* \in C^\infty(M)$ s.t. $f^*|_V = f|_V$ and $f^* \equiv 0$ outside of U.

Defn $D: C^\infty(M) \to C^\infty(M)$ is a linear differential operator if:

(i) $\forall$ open set $U \subset M$, $D(C^\infty(U)) \subset C^\infty(U)$

(ii) If $U \subset M$ open diffeo to $\mathbb{R}^n$, $\Rightarrow$ D has the form

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$$D(f) = \sum_{|\alpha| \le k} g_\alpha D^\alpha f = \sum_{|\alpha|\le k} g_\alpha \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}},$$

$g_\alpha \in \mathbb{R}$, $\alpha = (\alpha_1, \dots, \alpha_n) \in (\mathbb{N} \cup \{0\})^n$, $n = \dim M$, $|\alpha| = \sum_{i=1}^n \alpha_i$.

The order of D,

$$\text{ord}(D) := \max\{|\alpha| : g_\alpha \neq 0\}$$

Fact (1) ord (D) is indep. of the chart.

(2) $\text{DiffOp}(M)$ is an algebra under composition as product and

$$\text{ord}(D_1 D_2) \le \text{ord}(D_1) + \text{ord}(D_2)$$

Notation $C^\infty(p)$ = algebra of germs of smooth functions at p = algebra of smooth fcts. defn. in a nbd of p, where we identify two fcts if they coincide on a nbd.

Recall The tangent space T_pM to M at a pt p is the set all linear functionals

$$X_p : C^\infty(p) \longrightarrow \mathbb{R}$$ such

that $\forall \alpha, \beta \in \mathbb{R}$, $f, g \in C^\infty(p)$

- $X_p(\alpha f + \beta g) = \alpha X_p(f) + \beta X_p(g)$ (linearity)
- $X_p(fg) = (X_p f) g(p) + f(p) X(g)$ (Leibniz rule)

This linear funct. is called tangent vector to M at p

and T_pM is a vector space.

The tangent bundle to M is

$$TM = \bigcup_{p\in M} T_pM$$ and can

be made into a mfd

$(U \times \mathbb{R}^n, \varphi \times \psi)$, where (U, φ) is a coord chart at $p \in M$

$\psi : \mathbb{R}^n \longrightarrow T_pM$ is an isom.

$\Rightarrow \pi : TM \longrightarrow M$ is smooth.

Defn. A smooth vector field is a smooth section of the tangent bundle

$$X : M \longrightarrow TM$$

s.t. $\pi \circ X = id_M$.

In other words $X : M \to TM$, $p \to X_p \in T_pM$

s.t. $p \longmapsto X_p(f)$ is smooth for all $f \in C^\infty(M)$.

Remark (1) $f \in C^\infty(U)$, X v.f.

$$\Rightarrow Xf \in C^\infty(U)$$

(2) Xf at p, amounts to taking the derivative of f at p in the direction of X_p.

Prop $\text{Vect}(M)$ are all the linear differential operators of first order that assign 0 to constants.

Remark If $X, Y \in \mathrm{Vect}(M)$

$\Rightarrow$ in general

$\mathrm{ord}(XY) = 2 \Rightarrow XY \notin \mathrm{Vect}(M)$

$$\underset{\substack{\uparrow \\ \text{vector subspace} \\ \text{not a sub.}}}{\mathrm{Vect}(M)} \subset \underset{\substack{\uparrow \\ \text{algebra}}}{\mathrm{DiffOp}(M)}$$

However $\mathrm{ord}(XY - YX) = 1$

$\Rightarrow XY - YX \in \mathrm{Vect}(M)$.

Defn. A $\mathbb{K}$-algebra, $\mathbb{K} = \mathbb{R}, \mathbb{C}$

$\mathrm{End}(A) = \{\delta : A \to A : \delta$ preserves the $\mathbb{K}$-mod. str.$\}$

$= \{\delta : A \to A \quad \delta(\lambda a + \mu b) = \lambda \delta(a) + \mu \delta(b)\}$ $\lambda, \mu \in \mathbb{K}$, $a, b \in A$

$\delta \in \mathrm{End}(A)$ is called **derivation** if $\delta(ab) = \delta(a) b + a \delta(b)$ $\forall a, b \in A$.

$\mathrm{Der}(A)$ = deriv. of A.

Ex. $A = C^\infty(M)$ is an $\mathbb{R}$-algebra and

$\mathrm{Vect}(M) = \mathrm{Der}(C^\infty(M))$.

(Leibniz rule) ☒

In general $\mathrm{Der}(A)$ is a vector subspace but not a subalgebra

$$\delta_1 \delta_2 (ab) = \delta_1 \left(\delta_2(a) b + a \delta_2(b) \right)$$

$$= \delta_1 \delta_2(a) b + \delta_2(a) \delta_1(a) b + \delta_1(a) \delta_2(b) + a \delta_1 \delta_2(b)$$

$\Rightarrow (\delta_1 \delta_2 - \delta_2 \delta_1)(ab)$ is a derivation

$$(\delta_1 \delta_2 - \delta_2 \delta_1)(ab) = a (\delta_1 \delta_2 - \delta_2 \delta_1)(b) + (\delta_1 \delta_2 - \delta_2 \delta_1)(a)\, b$$

$$[\ ,\ ] : \mathrm{Der}(A) \times \mathrm{Der}(A) \longrightarrow \mathrm{Der}(A)$$

$$(\delta_1, \delta_2) \longmapsto [\delta_1, \delta_2] := \delta_1 \delta_2 - \delta_2 \delta_1$$

Properties: of $[\ ,\ ]$

(1) bilinear

(2) $[\delta_1, \delta_2] = -[\delta_2, \delta_1]$

(3) $[\delta_1, [\delta_2, \delta_3]] = [[\delta_1, \delta_2], \delta_3] + [\delta_2, [\delta_1, \delta_3]]$

Definition A **Lie algebra** $\mathfrak{g}$ is a vector space over $\mathbb{K}$ with a bilinear map

$[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called **bracket** such that

(1) $[X, Y] = -[Y, X]$

(2) $[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]]$.

Jacobi identity

Rk • $[\ ,\ ]$ is a non-ass. product. If it were associative

$$[X, [Y, Z]] = [[X, Y], Z]$$

• Jacobi identity

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$$

$$\delta_X : \mathfrak{g} \to \mathfrak{g}$$

$$\delta_X(Y) := [X,Y]$$

Jacobi identity $\equiv$ δ_X is a derivation

Examples (1) A algebra $\Rightarrow$

$\Rightarrow$ A is a Lie algebra

$$[a,b] := ab - ba$$

(2) $\mathrm{Vect}(M)$ is a Lie algebra with bracket

$$[X,Y]_p(f) = X_p(Yf) - Y_p(Xf)$$

Ex. Verify that this satisfies the Jacobi identity.

(3) $\mathbb{R}^{n\times n}$ Lie algebra

$$[A,B] = AB - BA$$

(4) Any vector space with $[\,,\,] \equiv 0$ is a Lie algebra.

Defn A Lie algebra $\mathfrak{g}$ is Abelian if $[X,Y]=0$ $\forall X,Y \in \mathfrak{g}$.

(5) V 2-dim. v.s. $V = \langle v, w \rangle$.

- $[v,v] = [w,w] = 0$
- $[v,w] := w$

$\Rightarrow$ Lie algebra

(6) $\mathbb{R}^3$ with crossproduct

(7) V 3-dim v.s. $V = \langle u, v, w \rangle$

$$\begin{cases} [u,u] = [v,v] = [w,w] = 0 \\ [u,v] = w \\ [u,w] = -2u \\ [v,w] = 2v \end{cases}$$

extend it by linearity.

Matrix realization

$$u = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\quad v = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\quad w = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Fast forward: $SL(2,\mathbb{R}) = \{ g \in GL(2,\mathbb{R}) : \det g = 1 \}$

$$\mathfrak{sl}(2,\mathbb{R}) = \{ X \in M_{2\times 2}(\mathbb{R}) : \mathrm{Tr}\, X = 0 \}$$

$\Rightarrow \left\langle \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle$ is $\mathfrak{sl}(2,\mathbb{R})$.

Defn A Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is a vector subspace s.t. $[X,Y] \in \mathfrak{h}$ $\forall X,Y \in \mathfrak{h}$.

Ex. • $\underset{\text{Lie (sub)algebra}}{\mathrm{Vect}(M)} \subset \underset{\text{subalgebra}}{\mathrm{DiffOp}(M)} \subset \underset{\text{algebra}}{\mathrm{End}(C^\infty(M))}$

• $\underset{\text{Lie (sub)algebra}}{\mathrm{Der}(A)} \subset \underset{\text{algebra}}{\mathrm{End}(A)}$

Recall $f : M \to N$ smooth map of smooth mfds, $p \in M$.

The differential of f at p is the linear map

$$d_p f : T_p M \longrightarrow T_{f(p)} M$$

defined as follows: if $X_p \in T_p M$ and $\varphi \in C^\infty(M)$, then

$$(d_p f)(X_p)(\varphi) := X_p(\varphi \circ f)$$

Defn : $\varphi : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homo if

$$\varphi\left([X,Y]_{\mathfrak{g}}\right) = [\varphi(X), \varphi(Y)]_{\mathfrak{h}}$$

Ex : $f : M \to M'$ diffeo.

$f_* : \mathrm{Vect}(M) \to \mathrm{Vect}(M')$

$$\begin{tikzcd} TM \arrow[r, "df"] & TM' \\ M \arrow[u, "X"] \arrow[r, "f"'] \arrow[ur, phantom, "\circlearrowright"] & M' \arrow[u, "f_*X"'] \end{tikzcd}$$

$$(f_*X)_{m'} = \left(d_{f^{-1}(m')} f\right) X_{f^{-1}(m')}$$

f_* is a Lie algebra homo

$$f_*([X,Y]) = [f_*X, f_*Y]$$

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