Exercise Class Lie Groups : 22.10.2020

Exercise 4.(Haar Measure and Transitive Actions):

Let G be a locally compact Hausdorff group and let X be a topological space. Suppose that G acts on X continuously and transitively. Let $o \in X$, and denote $\pi\colon G \to X, g \mapsto g \cdot o$. Further, let

$$H := \mathrm{Stab}(o) = \{h \in G \,|\, h \cdot o = o\}$$

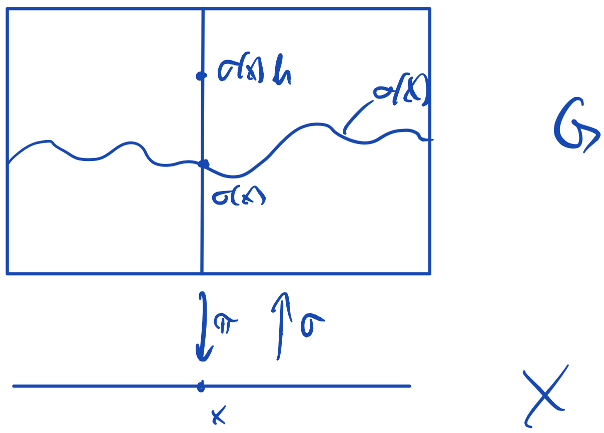
be the stabilizer of o.

Suppose there is a continuous section $\sigma\colon X \to G$ of π, i.e. $\pi \circ \sigma = \mathrm{Id}_X$.

a) Show that $\psi\colon X \times H \to G, (x,h) \mapsto \sigma(x)h$ is a homeomorphism.

Hint: Find a continuous inverse!

$$\begin{tikzcd} G \arrow[d, "\pi"] \\ X \arrow[u, bend right, dotted, "\sigma"'] \end{tikzcd}$$



Solution: Define $\varphi: G \longrightarrow X \times H$, $\varphi(g) = (\pi(g), \sigma(\pi(g))^{-1} g)$.

Why is φ well-def.; i.e. why is $\sigma(\pi(g))^{-1} g \in H$?

$$\sigma(\pi(g)) \cdot o = \cancel{\pi(\sigma}(\pi(g))) = \pi(g) = g \cdot o$$

$$\Rightarrow \underbrace{\sigma(\pi(g))^{-1} g}_{\in H} \cdot o = o$$

Check: (i) $\psi \circ \varphi = \mathrm{id}_G$ and (ii) $\varphi \circ \psi = \mathrm{id}_{X \times H}$.

(ii) $\varphi(\psi(x,h)) = \varphi(\sigma(x)h) = (\pi(\sigma(x)h), \underbrace{\sigma(\overbrace{\pi(\sigma(x)h)}^{=x})^{-1}}_{= \cancel{\sigma(x)^{-1}}} \cancel{\sigma(x)} h)$

$\sigma(x)h \cdot o = \sigma(x)o = \pi(\sigma(x)) = x$

$= (x, h)$ $\square$

b) Suppose there is a (left) Haar measure ν on H and suppose there is a left G-invariant Borel regular measure λ on X.

Show that the push-forward measure $\psi_*(\lambda\otimes\nu)$ is a (left) Haar measure on G.

Solution: Only thing to check: $\psi_*(\lambda\otimes\nu)$ is G-invariant.

Let $g_0\in G$ and let $f\in C_c(G)$.

$$\int_G f(g_0 g)\, d\psi_*(\lambda\otimes\nu)(g) = \int_{X\times H} f(g_0\,\psi(x,h))\, d(\lambda\otimes\nu)(x,h)$$

(Fubini) $$= \int_X\int_H f(g_0\,\sigma(x)h)\, d\nu(h)\, d\lambda(x)$$

$$= \int_X\int_H f\big(\sigma(g_0x)\underbrace{(\sigma(g_0x)^{-1}g_0\sigma(x))}_{\in H:\ \sigma(g_0x)\cdot o \,=\, g_0x \,=\, g_0\sigma(x)\cdot o}h\big)\, d\nu(h)\, d\lambda(x)$$

(ν H-inv.) $$= \int_X\int_H f(\sigma(g_0x)h)\, d\nu(h)\, d\lambda(x)$$

(G-inv. of λ) $$= \int_X\int_H f(\sigma(x)h)\, d\nu(h)\, d\lambda(x)$$

$$= \int_G f(g)\, d\psi_*(\lambda\otimes\nu)(g).$$

c) Find a Haar measure on $\mathrm{Iso}(\mathbb{R}^2)$.

Sol: $\mathrm{Iso}(\mathbb{R}^2) \curvearrowright \mathbb{R}^2$ transitively; section $\sigma: \mathbb{R}^2 \to \mathrm{Iso}(\mathbb{R}^2)$, $x \mapsto T_x$

translation by x: $T_x(y) = y + x$.

For $0 \in \mathbb{R}^2$, then σ is a section for $\pi: \mathrm{Iso}(\mathbb{R}^2) \to \mathbb{R}^2$, $g \mapsto g \cdot 0$

- $\mathrm{Stab}(0) = O_2(\mathbb{R})$ = orthogonal matrices
- Lebesgue measure λ on $\mathbb{R}^2$ is $\mathrm{Iso}(\mathbb{R}^2)$-invariant.

By part a) & b): For $f \in C_c(\mathrm{Iso}(\mathbb{R}^2))$

$$\int_{\mathrm{Iso}(\mathbb{R}^2)} f(g) \, d\varphi_*(\lambda \otimes \nu)(g) = \int_{\mathbb{R}^2} \int_{O_2(\mathbb{R})} f(T_x \cdot h) \, d\nu(h) \, d\lambda(x).$$

Applying b) again to $O_2(\mathbb{R}) \curvearrowright \{\pm 1\}$ we obtain

$$\int_{O_2(\mathbb{R})} h(k) \, d\nu(k) = \sum_{\varepsilon \in \{\pm 1\}} \int_0^{2\pi} h\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \varepsilon \cdot \sin(\theta) & \varepsilon \cdot \cos(\theta) \end{pmatrix} d\theta .$$

$\forall h \in C_c(O_2(\mathbb{R}))$.

All in all: $$\int_{\mathrm{Iso}(\mathbb{R}^2)} f(g) \, d\mu(g) = \int_{\mathbb{R}_2} \sum_{\varepsilon \in \{\pm 1\}} \int_0^{2\pi} f\left(T_x \circ \begin{pmatrix} \cos\theta & -\sin\theta \\ \varepsilon \cdot \sin\theta & \varepsilon \cdot \cos\theta \end{pmatrix}\right) d\theta \, d\lambda(x).$$

Other applications: $SL_2(\mathbb{R}) \curvearrowright \mathbb{H}^2 = \{x + iy \mid y > 0\} \subset \mathbb{C}$

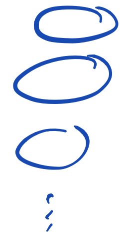
More generally: symmetric spaces, e.g. $SL_n(\mathbb{R}) / SO_n(\mathbb{R})$ (next semester)

CAVEAT: The section $\sigma : X \longrightarrow G$ does not need to exist!

Example: $\mathbb{R} \curvearrowright S^1 \subset \mathbb{C}$ via $t * \xi = e^{2\pi i t} \cdot \xi$

Then $\mathrm{Stab}(1) = \mathbb{Z}$

If there were a section $\sigma : S^1 \to \mathbb{R}$ then:

by a): $\mathbb{R} \cong S^1 \times \mathbb{Z} \cong \bigsqcup_{n \in \mathbb{Z}} S^1 =$ 

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5) c) Prove that there exists a discontinuous, bijective homomorphism from the additive group $(\mathbb{R}, +)$ to itself.

Sol: Take an infinite (Hamel) basis $B = \{x_i \mid i \in I\}$ of $\mathbb{R}$ over $\mathbb{Q}$ containing 1. Pick $i \neq j \in I$ and define $\varphi: \mathbb{R} \to \mathbb{R}$ by $\mathbb{Q}$-linear extension

$$\varphi(x_k) = \begin{cases} x_j, & \text{if } k = i \\ x_i, & \text{if } k = j \\ x_k, & \text{else} \end{cases}$$

Choose a sequ. $(q_n) \subset \mathbb{Q}$ conv. to x_i

Then $\lim\limits_{n\to\infty} \varphi(q_n) = \lim\limits_{n\to\infty} \varphi(q_n \cdot 1) = \lim\limits_{n\to\infty} q_n \cdot \underbrace{\varphi(1)}_{=1} = \lim\limits_{n\to\infty} q_n = x_i$

$\neq x_j = \varphi(x_i) = \varphi(\lim\limits_{n\to\infty} q_n)$

$\Rightarrow \varphi$ is not continuous!

Q: Is φ measurable?

A: No! By Mackey's thm. any meas. hom $\varphi: \mathbb{R} \to \mathbb{R}$ is already cont.

Exercise 7.(No $\mathrm{SL}_2(\mathbb{R})$-invariant Measure on $\mathrm{SL}_2(\mathbb{R})/P$):

Let $G = \mathrm{SL}_2(\mathbb{R})$ and P be the subgroup of upper triangular matrices. Show directly that there is no (non-trivial) finite G-invariant measure on G/P.

Hint: Identify $G/P \cong \mathbb{S}^1 \cong \mathbb{R} \cup \{\infty\}$ with the unit circle and consider a rotation

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

and a translation

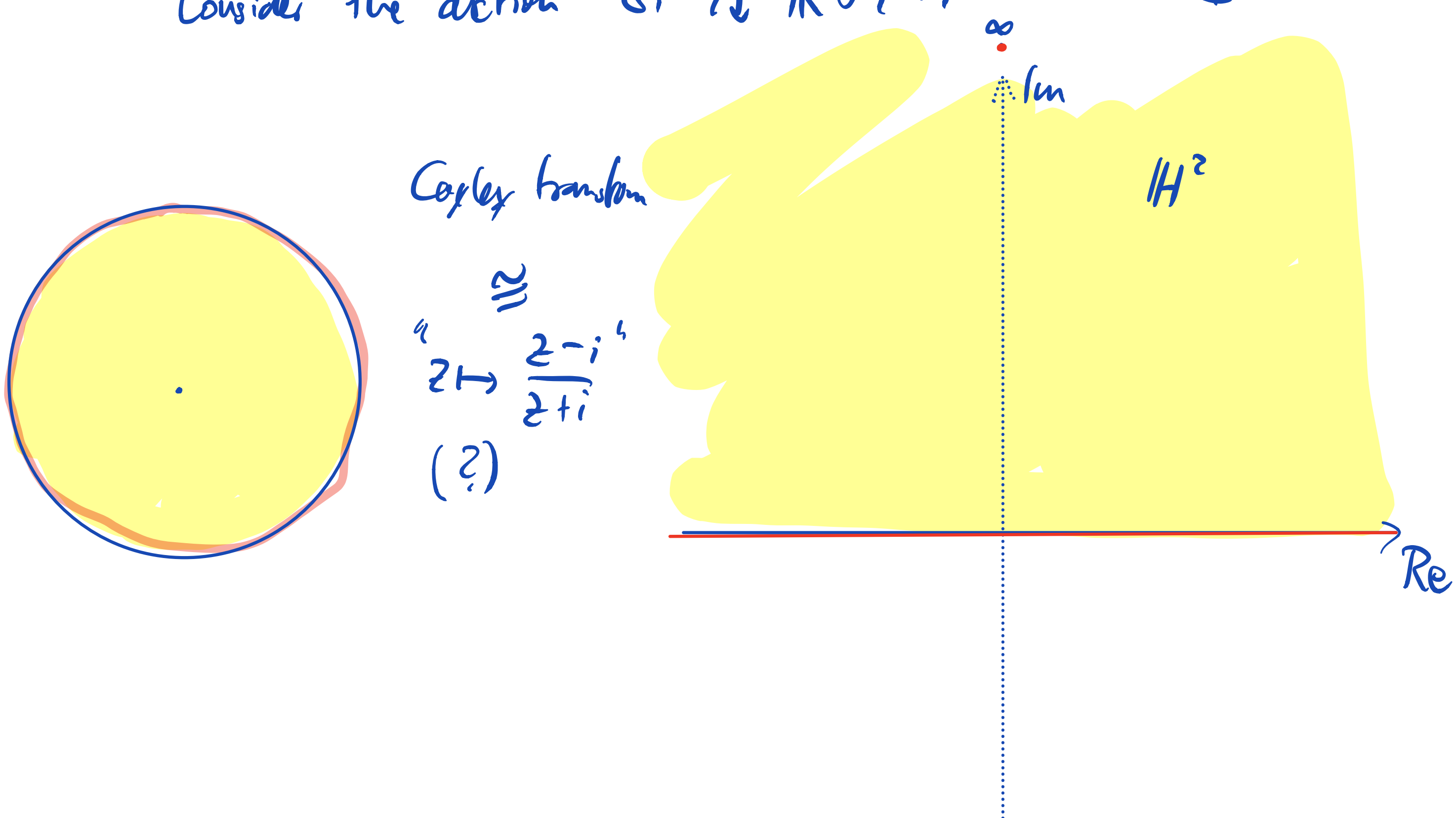
$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

Sol: Recall that $G = SL_2(\mathbb{R})$ acts via linear fractional transformations on $\mathbb{H}^2 = \{x+iy \mid y>0\}$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} . z = \frac{az+b}{cz+d}$$

This action extends cont. to the entire Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

Consider the action $G \curvearrowright \mathbb{R} \cup \{\infty\} = \partial \mathbb{H}^2 \cong \mathbb{S}^1$



Note: $\mathcal{U} = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \,\middle|\, t \in \mathbb{R} \right\} \subset SL_2(\mathbb{R})$ and

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} . x = \frac{x + t}{0 + 1} = x + t.$$

Suppose that there is a non-triv. f. measure m on $\mathbb{R} \cup \{\infty\}$.
Then $m|_{\mathbb{R}}$ is also finite & $\mathcal{U}$-invariant; i.e. translation invariant. $\Longrightarrow$ $m|_{\mathbb{R}}$ is a multiple of Lebesgue measure $\Longrightarrow$ $m|_{\mathbb{R}} \equiv 0$.

There is $i := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in SL_2 \mathbb{R}$, and

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} . \infty = 0$$

$\Longrightarrow$ For any sufficiently small nbhd U of ∞:

$i(U) \subset \mathbb{R}$ but

$$m(U) = m(\underbrace{i(U)}_{\subset \mathbb{R}}) \equiv 0.$$

$\Longrightarrow$ $m \equiv 0$ (hence trivial).

This is not a contradiction to the Theorem from class:

<u>Thm:</u> G l.c. H. group, $H \leq G$ closed subgroup. Then $\exists$ a positive invariant measure on G/H if and only if $\Delta_G|_H = \Delta_H$.

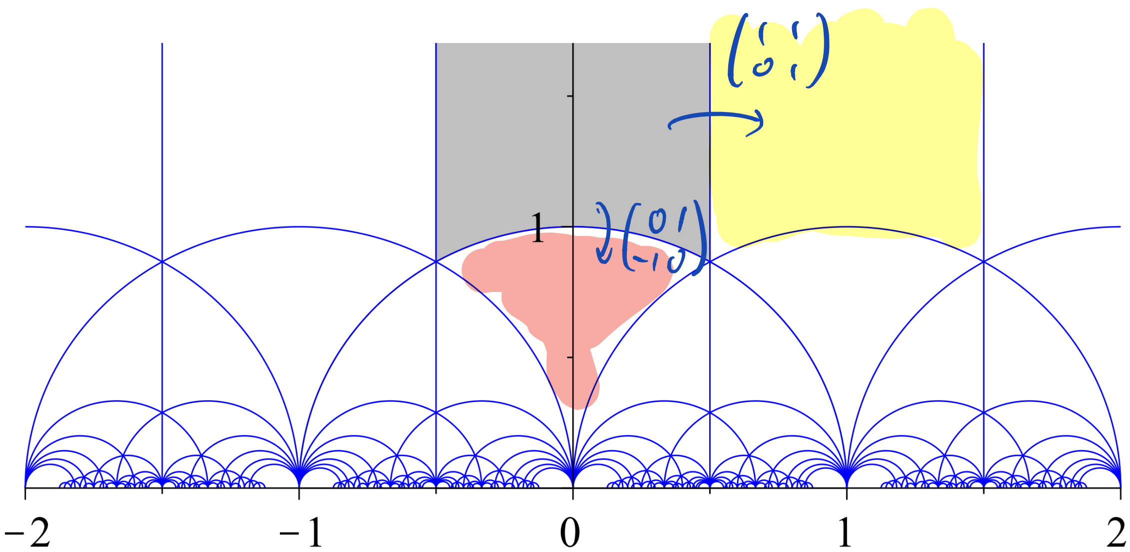
<u>Indeed:</u>
- By Ex. 3 d), $SL_2(\mathbb{R})$ is unimodular; i.e. $\Delta_{SL_2(\mathbb{R})} \equiv 1$
- By Ex. 3 b), $P = \left\{\begin{pmatrix} a & b \\ 0 & a^{-1}\end{pmatrix} \in SL_2\mathbb{R}\right\}$ has $\frac{da}{a^2}\,db$ as a LEFT Haar measure, but $da\,db$ as a RIGHT Haar measure.

$\Longrightarrow$ P is not unimodular, i.e $\exists\, p \in P$: $\Delta_P(p) \neq 1 = \Delta_G(p)$.

LATTICES & UNIMODULARITY

<u>Recall</u>: Let G be a locally compact Hausdorff group.
A <u>lattice</u> $\Gamma \leq G$ is a discrete subgroup s.t. there exists a FINITE G-invariant measure on the quotient G/Γ.

<u>Ex</u>: 1) $\mathbb{Z}^n < \mathbb{R}^n$, 2) $SL_2(\mathbb{Z}) < SL_2(\mathbb{R})$



<u>Lemma</u>: Let G be a locally compact Hausdorff group and $\Gamma \leq G$ a lattice. Then G is unimodular, i.e. $\Delta_G \equiv 1$.

<u>Proof</u>: $\Delta_G|_\Gamma \equiv \Delta_\Gamma \equiv 1$ (by Thm. about quot. m.)

$$\Longrightarrow \quad \begin{tikzcd} & G \arrow[dl, "\pi"'] \arrow[dr, "\Delta_G"] & \\ G/\Gamma \arrow[rr, "\bar{\Delta}_G"] & & \mathbb{R}_{>0} \end{tikzcd}$$

$\bar{\Delta}_G(g\Gamma) := \Delta_G(g)$

Push-forward the finite inv. measure $\nu_{G/\Gamma}$ from G/Γ to a finite measure $\nu^* := (\bar{\Delta}_G)_*(\nu_{G/\Gamma})$ on $\mathbb{R}_{>0}$.

Because G/Γ is G-invariant, ν^* is $\Delta_G(G)$-invariant.

$$\int_{\mathbb{R}_{>0}} f(\Delta_G(g)\cdot x)\, d(\overline{\Delta}_G)_*(\nu_{G/\Gamma})(x) = \int_{G/\Gamma} f(\underbrace{\Delta_G(g)\cdot\Delta_G(h)}_{=\Delta_G(gh)=\overline{\Delta}_G(gh\Gamma)})\, d\nu_{G/\Gamma}(h\Gamma)$$
$$= \int_{G/\Gamma} f(\overline{\Delta}_G(h\Gamma))\, d\nu_{G/\Gamma}(h\Gamma) \qquad \square$$

Claim: There is no non-trivial finite measure ν on $\mathbb{R}_{>0}$ that is invariant under translation by some $t\in\mathbb{R}_{>0}\setminus\{1\}$.

Use the claim, to argue as follows:

Suppose $\Delta_G(G)\neq\{1\}$. Then $\exists\, t\in\mathbb{R}_{>0}\setminus\{1\}: t\in\Delta_G(G)$

But ν^* is t-invariant!

non-triv. finite measure.

Claim $\Rightarrow$ $\nu^*\equiv 0$ $\lightning$ $\nu^* = (\overline{\Delta}_G)_*(\nu_{G/\Gamma})$

Proof of claim: Let $t\in\mathbb{R}_{>0}\setminus\{1\}$ s.t. a f. m. ν on $\mathbb{R}_{>0}$ is t-inv.

Look at intervalls of the form $[x\cdot t^n, x\cdot t^{n+1})$ for $x\in\mathbb{R}$.

$$\nu([x\cdot t^n, x\cdot t^{n+1})) = \nu(t\cdot[x\cdot t^n, x\cdot t^{n+1}))$$
$$= \nu([x\,t^{n+1}, x\cdot t^{n+2}))$$

and $\mathbb{R}_{>0} = \bigsqcup_{n\in\mathbb{Z}} [x\cdot t^n, x\cdot t^{n+1})$

$$+\infty > \sigma(\mathbb{R}_{>0}) = \sum_{n\in\mathbb{Z}} \underbrace{\sigma([x\cdot t^n, x\cdot t^{n+1}))}_{= \sigma([x, x\cdot t))}$$

$$\Longrightarrow \quad \sigma([x\cdot t^n, x\cdot t^{n+1})) \equiv 0.$$

But $\{[x\cdot t^n, x\cdot t^{n+1})\}$ generate the Borel σ-alg.

$$\Longrightarrow \quad \sigma \equiv 0.$$

$\square$