28 October 2020

Last week (1) $\mathrm{Vect}(G)^G \cong T_e G$

$G \curvearrowright M$ transitive but not free

$\mathrm{Vect}(M)^G \cong T_{m_0} M^{\rho(G_{m_0})}$, where

$\rho : G_{m_0} \longrightarrow GL(T_{m_0} M)$

(2) If $G = GL(n,\mathbb{R})$ then

$T_I GL(n,\mathbb{R}) \overset{(*)}{\cong} \mathbb{R}^{n\times n}$

Thm $\mathrm{Vect}(GL(n,\mathbb{R}))^{GL(n,\mathbb{R})} \cong T_I GL(n,\mathbb{R})$ as Lie algebras, where $T_I GL(n,\mathbb{R})$ has the Lie alg. structure given by $(*)$.

Today Same holds for a subgp of $GL(n,\mathbb{R})$.

Rk G, H Lie gps, $\varphi : G \to H$ Lie gp homo $\Rightarrow \varphi(e_G) = e_H$

$\Rightarrow d_e\varphi : \mathfrak{g} \longrightarrow \mathfrak{h}$

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Proposition If $\varphi : G \to H$ is a Lie gp. homo $\Rightarrow d_e\varphi : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homo.

Corollary If $G \leq H$ is an inclusion of Lie gps $\Rightarrow T_e G \hookrightarrow T_e H$ is an inclusion of vg spaces that defines a Lie algebra embedding $\mathfrak{g} \hookrightarrow \mathfrak{h}$. Thus if $G \leq GL(n,\mathbb{R})$ the bracket on $\mathfrak{g}$ is the bracket on $\mathfrak{gl}(n,\mathbb{R}) \cong \mathbb{R}^{n\times n}$.

Remark 1 M, M' smooth mfds, $f : M \to M'$ smooth map,

$X \in \mathrm{Vect}(M) \Rightarrow d\varphi(X)$ is not (necessarily) a v.f. $\mathrm{Vect}(M')$.

$$\begin{tikzcd} TM \arrow[r, "d\varphi"] & TM' \\ M \arrow[u, "X"] \arrow[r, "f"] & M' \arrow[u] \end{tikzcd}$$

Two possible pbs:

(1) f not nec. onto

$$X_{m'} := d_{f^{-1}(m')} f \, X_{f^{-1}(m')}$$

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(2) There could be $p, q \in M$, $p \neq q$ s.t. $f(p) = f(q)$ and

$$(d_p f)(X_p) \neq (d_q f)(X_q)$$

Defn. $f : M \to M'$ smooth map, $X \in \mathrm{Vect}(M)$, $X' \in \mathrm{Vect}(M')$ are *f-related* if the diagram commutes, that is

$$X' \circ f = df \circ X \qquad \begin{tikzcd} TM \arrow[r, "df"] & TM' \\ M \arrow[u, "X"] \arrow[r, "f"] \arrow[ur, phantom, "\circlearrowleft"] & M' \arrow[u, "X'"'] \end{tikzcd}$$

Exercise If $f : M \to M'$ smooth, $X_i \in \mathrm{Vect}(M)$, $X_i' \in \mathrm{Vect}(M')$ are v.f. with X_i f-related to X_i', $i = 1,2$, $\Rightarrow$

$\Rightarrow [X_1, X_2]$ is f-related to $[X_1', X_2']$.

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That is

$[X_1', X_2'] \circ f = df \circ [X_1, X_2]$,

i.e. $\forall m \in M$

$$[X_1', X_2']_{f(m)} = (d_m f)([X_1, X_2]_m)$$

Remark 2 A vector field $X \in \mathrm{Vect}(G)$ is left invariant iff it is L_g-related to itself. In fact

X L_g-related to itself $\Longleftrightarrow$

$\Longleftrightarrow X \circ L_g = dL_g \circ X \Longleftrightarrow$

$\Longleftrightarrow \forall h \in G \quad X_{gh} = d_h L_g X_h$

Take $h = e$.

Remark 3 $\varphi : G \to H$ homo

$\varphi(gh) = \varphi(g)\varphi(h) \Rightarrow$

$\varphi(L_g h) = L_{\varphi(g)} \varphi(h) \Rightarrow$

$\Rightarrow \varphi \circ L_g = L_{\varphi(g)} \circ \varphi$

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Proof of proposition

We want to show that

$$d_e\varphi([X,Y]) = [d_e\varphi(X), d_e\varphi(Y)],$$

where $X, Y \in \mathfrak{g} \simeq T_eG$.

$X \in \mathfrak{g} \rightsquigarrow d_e\varphi(x) \in \mathfrak{h}$ and each of them defines a left inv. v.f. on G and H respectively.

$\tilde{X} \in (\mathrm{Vect}(G))^G \qquad (\tilde{X})_{e_G} = X$

$\bar{X} \in \mathrm{Vect}(H)^H \qquad \bar{X}_{e_H} = (d_{e_G}\varphi)(X)$

Claim $\tilde{X}$ and $\bar{X}$ are φ-related.

If so, by the exercise, if $\tilde{X}, \bar{X}$ φ-related
$\tilde{Y}, \bar{Y}$ φ-related

$\Rightarrow [\tilde{X}, \tilde{Y}]_{\mathfrak{g}}$ φ-rel. to $[\bar{X}, \bar{Y}]_{\mathfrak{h}}$, that is

$d\varphi\, [\tilde{X}, \tilde{Y}] = [\bar{X}, \bar{Y}] \circ \varphi$, i.e. $\forall g \in G$

$$d_g\varphi([\tilde{X},\tilde{Y}]_g) = [\bar{X},\bar{Y}]_{\varphi(g)}$$

In particular for $g = e_G$

$$d_{e_G}\varphi(\underbrace{[\tilde{X},\tilde{Y}]_{e_G}}_{=[X,Y]}) = [\bar{X},\bar{Y}]_{e_H}$$

$$d_e\varphi([x,Y]) = [d_e\varphi X, d_e\varphi Y]$$

Pf of claim

$$\bar{X} \circ \varphi(g) = \bar{X}_{\varphi(g)} \overset{\text{inv of } \bar{X}}{=} d_{\varphi(g)}L_{\varphi(g)}\bar{X}_{e_H}$$

$$\overset{\text{defn.}}{=} (d_{\varphi(g)}L_{\varphi(g)})(d_{e_G}\varphi)(X) =$$

$$= d_{e_G}(L_{\varphi(g)} \circ \varphi)(X)$$

$$= d_{e_G}(\varphi \circ L_g)\, X_{e_G}$$

$$= (d_g\varphi)(d_{e_G}L_g)\, X_{e_G} =$$

$$= (d_g\varphi)(\tilde{X})_g. \qquad \blacksquare$$

Example $SL(n,\mathbb{R}) = \{g \in GL(n,\mathbb{R}) : \det g = 1\}$.

To compute $\mathfrak{sl}(n,\mathbb{R})$ we take $\gamma : (-\varepsilon,\varepsilon) \longrightarrow SL(n,\mathbb{R})$ with $\gamma(0) = I$ and compute

$$0 = \frac{d}{dt}\Big|_{t=0} \det\gamma(t) =$$

$$= \underbrace{d_I\det}_{\mathrm{tr}} \circ\, \gamma'(0) = \mathrm{tr}\,\gamma'(0)$$

$$\Rightarrow \mathfrak{sl}(n,\mathbb{R}) \subseteq \{A \in \mathfrak{gl}(n,\mathbb{R}) : \mathrm{tr}\,A = 0\}$$

By dimension reasons we have equality.

Lemma Let $A, B : (-\varepsilon,\varepsilon) \longrightarrow \mathbb{R}^{n\times n}$ smooth curves and $\varphi(t) = A(t)B(t) \Rightarrow$

$$\Rightarrow \varphi'(t) = A'(t)B(t) + A(t)B'(t).$$

Example

$$O(n,\mathbb{R}) = \{g \in GL(n,\mathbb{R}) : {}^tgg = \mathrm{Id}\}$$

Let $\gamma : (-\varepsilon,\varepsilon) \longrightarrow O(n,\mathbb{R})$, i.e. ${}^t\gamma(t)\gamma(t) = \mathrm{Id} \Rightarrow$

$$\Rightarrow 0 = \frac{d}{dt}\Big|_{t=0} {}^t\gamma(t)\gamma(t) =$$

$$= ({}^t\gamma(t)'\gamma(t) + {}^t\gamma(t)\gamma'(t))\big|_{t=0} =$$

$$= {}^t\gamma(0)' + \gamma'(0) \Rightarrow$$

$\Rightarrow \mathfrak{o}(n,\mathbb{R}) = \mathrm{Lie}(O(n,\mathbb{R})) \subseteq$

$\subseteq \{A \in \mathfrak{gl}(n,\mathbb{R}) : {}^tA + A = 0\}$

By dimension counting $\Rightarrow =$.

Example $O(p,q) = \{g \in GL(p+q,\mathbb{R}) : gJ{}^tg = J\}$ where

$$J = \left(\begin{array}{c|c} -I_p & 0 \\ \hline 0 & I_q \end{array}\right) \Rightarrow$$

$\Rightarrow \mathfrak{o}(p,q) = \mathrm{Lie}(O(p,q)) =$

$= \{A \in \mathfrak{gl}(p+q,\mathbb{R}) : AJ + J{}^tA = 0\}$

Example $N = \left\{\begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \in GL(n,\mathbb{R})\right\}$

$\Rightarrow \mathfrak{n} = \left\{\begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix} \in \mathfrak{gl}(n,\mathbb{R})\right\}$

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Example $A_{\mathrm{diag}} = \left\{\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} : \lambda_i \neq 0\right\}$

$\mathfrak{a}_{\mathrm{diag}} = \left\{\begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_n \end{pmatrix} : \alpha_j \in \mathbb{R}\right\}$.

Example $U(n) = \{g \in GL(n,\mathbb{C}) : g{}^t\bar{g} = \mathrm{Id}\}$.

$\mathfrak{u}(n) = \{A \in \mathfrak{gl}(n,\mathbb{C}) : A + {}^t\bar{A} = 0\} =$

$=$ skew-hermitian matrices.

Example $B : \mathbb{C}^{2n} \times \mathbb{C}^{2n} \to \mathbb{C}$

skew-symm. bilinear form

$$B(x,y) = \sum_{p=1}^{n} (x_p y_{n+p} - x_{n+p} y_p)$$

$x = (x_1, \dots, x_{2n})$, $y = (y_1, \dots, y_{2n})$

Symplectic group

$Sp(2n,\mathbb{C}) := \{g \in GL(2n,\mathbb{C}) :$

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$B(x,y) = B(gx, gy)\} =$

$= \{g \in GL(2n,\mathbb{C}) : {}^tgFg = F\}$,

where $F = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$

$Sp(2n,\mathbb{R}) := Sp(2n,\mathbb{C}) \cap GL(2n,\mathbb{R})$

$Sp(2n) := Sp(2n,\mathbb{C}) \cap U(2n)$

is a compact gp.

$\mathfrak{sp}(2n,\mathbb{C}) = \{A \in \mathfrak{gl}(2n,\mathbb{C}) : {}^tAF + FA = 0\}$

$\mathfrak{sp}(2n,\mathbb{R}) = \{A \in \mathfrak{gl}(2n,\mathbb{R}) : {}^tAF + FA = 0\}$.

Real Lie gps

$SL(n,\mathbb{R}) > SO(n,\mathbb{R})$

$SO(p,q) > S(O(p) \times O(q))$

$SU(p,q) > S(U(p) \times U(q))$

$Sp(2n,\mathbb{R}) > U(n) = O(2n,\mathbb{R}) \cap Sp(2n,\mathbb{R})$

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Complex Lie gps

$GL(n,\mathbb{C}) > U(n)$

$SL(n,\mathbb{C}) > SU(n) = U(n) \cap SL(n,\mathbb{C})$

$O(n,\mathbb{C}) > SO(n,\mathbb{R}) = SL(n,\mathbb{R}) \cap O(n,\mathbb{C})$

$Sp(2n,\mathbb{C}) > Sp(2n) = Sp(2n,\mathbb{C}) \cap U(2n)$.

- G, H Lie gps $\varphi : G \to H$ homo $\Rightarrow d_e\varphi : \mathfrak{g} \to \mathfrak{h}$ Lie alg. homo.

Inverse statement should be interpreted correctly.

Two diff. kinds of results:

1) If G is a Lie gp with Lie algebra $\mathfrak{g}$, and $\mathfrak{h} \subset \mathfrak{g}$ is a Lie subalg.

$\overset{?}{\Rightarrow} \exists H \leq G$ s.t.

$\mathfrak{h} = \mathrm{Lie}(H)$?

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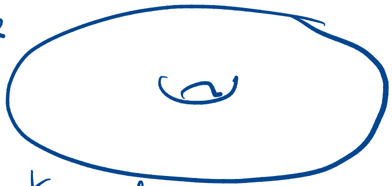
2) Given G, H Lie gps with Lie algebras $\mathfrak{g}, \mathfrak{h}$, given $\pi : \mathfrak{g} \to \mathfrak{h}$ Lie alg. homo,

$\overset{?}{\Longrightarrow}$ Is there $\varphi : G \to H$ s.t. $d_e\varphi = \pi$?

The answer to (1) is yes, with conditions.

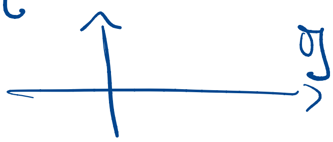
Example $G = \mathbb{T}^2 \simeq S^1 \times S^1$

$\mathfrak{g} = \mathbb{R}^2$ because

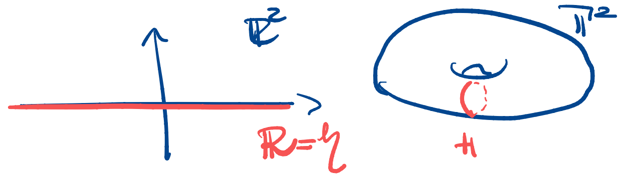


($\mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2$ has discrete kernel).

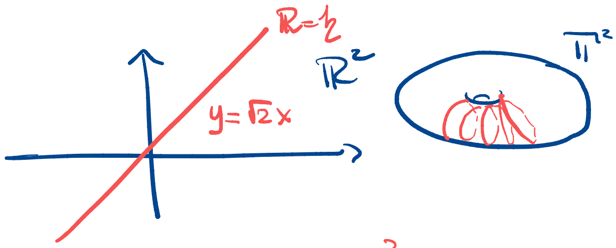
$\mathfrak{h} = \mathbb{R}$ Lie subalgebra.



(1)



(2)



$\varphi : \mathbb{R} \longrightarrow \mathbb{R}^2$

$t \longmapsto (e^{it}, e^{i\sqrt{2}t})$

Theorem Given G Lie gp with Lie algebra $\mathfrak{g}$ and a Lie subalgebra $\mathfrak{h} \subset \mathfrak{g} \Rightarrow \exists !$ immersed subgroup H s.t. $\mathrm{Lie}(H) = \mathfrak{h}$.