29 October 2020

G Lie gp with $Lie(G) = \mathfrak{g}$, $\mathfrak{h} \subset \mathfrak{g}$ Lie subalgebra $\Rightarrow$ $\exists$ 1-1 immersed subgp H s.t. $Lie(H) = \mathfrak{h}$.

Will follow from Frobenius' thm. relating v.f. on a mfd to submanifolds.

Defn. $\varphi : M \to N$ smooth map of smooth mfds:

(1) φ is an *immersion* if the differential is non-singular $\forall\, p \in M$.

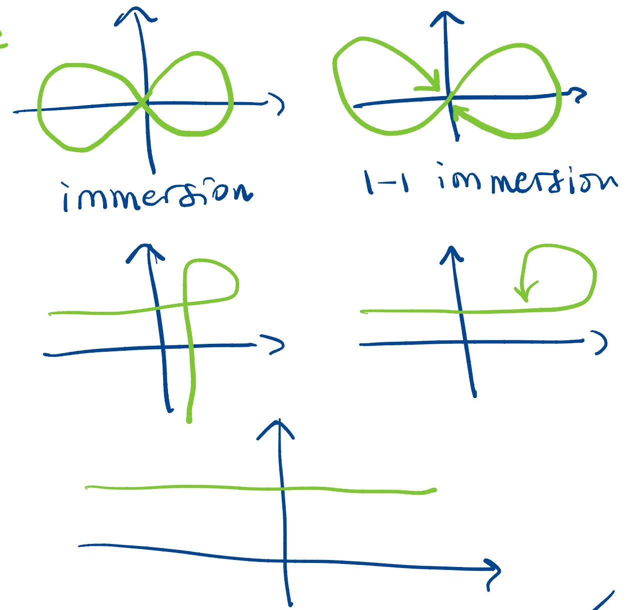
(2) φ is a *1-1 immersion* if it is an immersion & 1-1, in which case $\varphi(M)$ is an *immersed submanifold* or a *submanifold*.

Rk Regular submanifold has the regular submfd property, /1

that is every pt has a coord. nbd. with respect to which the subm. is defn by $x_{k+1} = \dots = x_n = 0$

Equivalently φ is a 1-1 immersion that is also a homeo $\Rightarrow$ φ is an *embedding* and $\varphi(M)$ is a regular submanifold.

Ex



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Theorem

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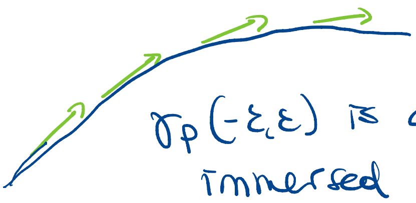
Example M smooth mfd, $p \in M$, $X \in Vect(M) \Rightarrow \exists$

$\gamma_p : (-\varepsilon, \varepsilon) \longrightarrow M$ s.t.

$\gamma_p(0) = p$, $\gamma_p'(t) = X_{\gamma_p(t)}$



$\gamma_p(-\varepsilon, \varepsilon)$ is a 1-1 immersed submfd whose tangent space is spanned by X.

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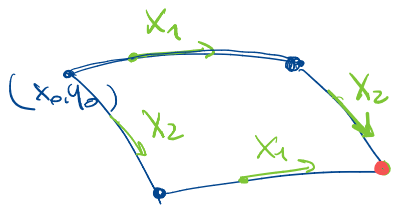
Example
$$\begin{cases} \dfrac{\partial z}{\partial x} = G_1(x,y,z) \\ \dfrac{\partial z}{\partial y} = G_2(x,y,z) \end{cases}$$

$z = z(x,y)$ sol. in a nbd of $(x_0,y_0) = z_0$. $\Rightarrow$ A solution is a function $z = f(x,y)$ s.t.

$$\begin{cases} f_x = G_1(x,y,f(x,y)) \\ f_y = G_2(x,y,f(x,y)) \end{cases}$$

Finding a solution is finding a surface whose tg space at (x,y) is spanned by

$$\begin{cases} X_1 = (1, 0, G_1(x,y,f(x,y))) \\ X_2 = (0, 1, G_2(x,y,f(x,y))) \end{cases}$$



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Defn. (1) Let M be a mfd of dim. $m=n+k$ and for each $p\in M$, let $\mathcal{D}_p \subset T_pM$ be an n-dim. subspace of T_pM. We say that $\mathcal{D}$ is a smooth distribution of dimension n if ~~any nbd~~ ~~$\forall$ of~~ for any pt $p\in M$ there exists nbd U & smooth vector fields $X_1,\dots,X_n$, that are a basis of $\mathcal{D}_q$ for every $q\in U$.

(2) A smooth distribution is involutive if $\exists$ a local basis $\{X_1,\dots,X_n\}$ of $\mathcal{D}$ such that $[X_i,X_j]\in\mathcal{D}$ $\forall\, i,j=1,\dots,n$

(3) If $\mathcal{D}$ is a smooth distr. and $\varphi: N\to M$ is a 1-1 immersion, then $\varphi(N)$ is

an integral submanifold of $\mathcal{D}$ if $(d_p\varphi)T_pN = \mathcal{D}_{\varphi(p)}$.

(4) We say that $\mathcal{D}$ is completely integrable if $\exists$ an integral submanifold.

Proposition Any completely integrable distribution is involutive.

Example (1) $M=\mathbb{R}^n\times\mathbb{R}^k$,

$X_i=\frac{\partial}{\partial x_i}$, $i=1,\dots,n$ $\Rightarrow$

$\Rightarrow \mathcal{D}=\{\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n}\}$ is an involutive distribution.

(2) G Lie gp. with $\mathrm{Lie}(G)=\mathfrak{g}$, $\mathfrak{h}\subset\mathfrak{g}$ Lie subalgebra $\Rightarrow$

$\Rightarrow$ $\mathfrak{h}$ is an involutive distb.

3) $M=\mathbb{R}^3$, $\mathcal{D}=\{\frac{\partial}{\partial x},\frac{\partial}{\partial y}+x\frac{\partial}{\partial z}\}$ is not involutive since

$$\left[\frac{\partial}{\partial x},\frac{\partial}{\partial y}+x\frac{\partial}{\partial z}\right]=\frac{\partial}{\partial z}.$$

Theorem (Frobenius) A smooth distribution is involutive iff it is completely integrable.

Remark on the proof

Involutive allows to reduce a system of PDEs into a system of ODEs for which we know that there is a solution.

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Defn. A maximal integral submanifold of a distrib $\mathcal{D}$ is a connected integral submfd that is not a proper subset of any other conn. integral submfd, That is it contains any other integral submfd with which it shares a pt.

Thm Given an involutive distrib. $\exists$ always a unique maximal integral submanifold through a given point.

[Warner "Found. of diff. mfds & Lie gps" Springer]

[(Frobenius) Shigeyuki Morita "Geom. of diff. forms" AMS 2001]

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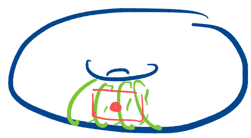
To prove the thm:

(1) If $p \in M$ $\exists$ coord. nbd (U, φ), $U = (-\varepsilon, \varepsilon)^{n+k}$ centered at p s.t. the integral submfd has the shape $x_{n+i} = \text{constant}$ for $i = 1, \dots, k$.

in the nbd U one can have many slices of the 1-1 immersed submanifold



Proposition Let $\mathcal{D}$ be an invol. distribution on M and let N be a max. integral submfd. If $f: M' \to M$ is a smooth map of smooth mfds and $f(M') \subset N \Rightarrow f: M' \to N$ is smooth.

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Proof of theorem about Lie subalgebras.

Let $X_1, \dots, X_n$ be left inv. v.f. on G s.t. $\operatorname{span}\{X_1, \dots, X_n\} = \mathfrak{h}$. Since $\mathfrak{h}$ is a Lie subalgebra $\Rightarrow \mathcal{D} = \{X_1, \dots, X_n\}$ is involutive $\Rightarrow$ by Frobenius thm $\exists$ integral submanifold. In fact $\exists$ a maximal integral submanifold H through $e \in G$. Since $X_i \in \mathrm{Vect}(G)^G \Rightarrow L_g H$ is also a maximal integral submanifold $\forall g \in G$. Thus in particular $L_{h^{-1}} H$ is a m.i.s. through e if $h \in H$.

$\Rightarrow L_{h^{-1}} H = H \quad \forall h \in H \Rightarrow$

$\Rightarrow H$ is a subgroup.

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The maps $m: H \times H \to G$ and $\mathrm{inv}: H \to G$ are also smooth as fctn into H by the previous proposition $\Rightarrow$ done.

The uniqueness of H comes from the uniqueness of max. integral submanifolds. $\boxtimes$

(1) $\varphi: G \to H$ homo of Lie gps $\Rightarrow$

$\Rightarrow d_e\varphi: \mathfrak{g} \to \mathfrak{h}$ homo of Lie algebras.

(2) Can we have a converse?

(i) Thm just proved

(ii) Q: If G, H are Lie gps, and $\pi: \mathfrak{g} \to \mathfrak{h}$ is a homo of their Lie algebras, does there exist $\varphi: G \to H$ s.t. $d_e\varphi = \pi$?

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Example $\varphi: \mathbb{R} \to S^1$ homo
$t \mapsto e^{it}$

$d_0\varphi: \mathrm{Lie}(\mathbb{R}) \to \mathrm{Lie}(S^1)$ isom.

$\Rightarrow (d_0\varphi)^{-1}: \mathrm{Lie}(S^1) \to \mathrm{Lie}(\mathbb{R})$

$\Rightarrow \nexists$ homo $\varphi: S^1 \to \mathbb{R}$ since $\nexists$ cpt 1-dim. sbgps of $\mathbb{R}$. So the answer to the question is NO, but $\exists$ always local homomorphism

Defn. (1) Let G, H be top. gps. A local homomorphism is a continuous map

$\varphi: U \to H$, where $U \ni e_G$ is an open nbd of e_G, such that $\forall x, y \in U$ s.t. $xy \in U$ then $\varphi(xy) = \varphi(x)\varphi(y)$

(2) A local homo is a local isomorphism if it is bijective onto $\varphi(U)$ and $\varphi^{-1}: \varphi(U) \to G$ is continuous.