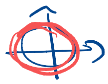
04 November 2020

Goal: G, H Lie gps, $\mathrm{Lie}(G) = \mathfrak{g}$, $\mathrm{Lie}(H) = \mathfrak{h}$, $\pi : \mathfrak{g} \to \mathfrak{h}$ homo of Lie algebras $\overset{?}{\Rightarrow}$ $\exists\, \varphi : G \to H$ Lie gps homo s.t. $d_e\varphi = \pi$?

Ex. $\varphi : \mathbb{R} \to S^1$



$\mathrm{Lie}(\mathbb{R}) \xrightarrow{\cong} \mathrm{Lie}(S^1)$

$\mathrm{Lie}(S^1) \longrightarrow \mathrm{Lie}(\mathbb{R})$

$\nexists\, \varphi : S^1 \to \mathbb{R}$

Answer is no.

Defn. (1) A local homo of top. gps G, H is a cont. map $\varphi : U \to H$ where $U \ni e_G$ nbd of $e_G \in G$ and s.t. $\forall\, x, y \in U$ s.t. $xy \in U \Rightarrow$

$\Rightarrow \varphi(xy) = \varphi(x)\varphi(y)$

(2) A local isom. is a local homo s.t. φ is bijective onto $\varphi(U)$ and $\varphi^{-1} : \varphi(U) \to U$

is continuous.

Remark For Lie gps replace cont. with smooth.

Thm (1) G, H Lie gps, $\mathrm{Lie}(G) = \mathfrak{g}$, $\mathrm{Lie}(H) = \mathfrak{h}$, $\pi : \mathfrak{g} \to \mathfrak{h}$ homo of Lie algebras. Then $\exists\, \varphi : U \to H$ Lie gps local homo s.t. $d_e\varphi = \pi$?

(2) If π is a Lie algebra isom $\Rightarrow \varphi$ is a local isom.

Lemma If $\varphi : U \to H$ is a local homo of Lie gps and $d_e\varphi : \mathfrak{g} \to \mathfrak{h}$ is an isom of Lie algebras $\Rightarrow \varphi$ is a local isom.

Pf Let φ be defnd on $U \subset G$. Since $d_e\varphi$ is bijective, by the Inv. F. Thm. $\exists\, e_G \in U' \subset G$ and $e_H \in V \subset H$ s.t. $\varphi : U' \to V$ is a diffeo.

$\Rightarrow \varphi : U \cap U' \to H$ is a local isom. $\boxtimes$

Pf of Thm (1)

Important point: Since π is a Lie algebra homo $\Rightarrow$ $\mathrm{graph}(\pi) \subset \mathfrak{g} \times \mathfrak{h}$ is a Lie subalgebra.

$[(X, \pi(X)), (Y, \pi(Y))] =$

$= ([X, Y], [\pi(X), \pi(Y)])$

$= ([X, Y], \pi([X, Y]))$.

$\Rightarrow \exists\, K < G \times H$ an immersed subgp s.t. $\mathrm{Lie}(K) = \mathrm{graph}(\pi)$

$\mathrm{Graph}(\pi) \hookrightarrow \mathfrak{g} \times \mathfrak{h} \xrightarrow{d\,pr_G} \mathfrak{g}$

$$\begin{tikzcd} K \arrow[r, hook] \arrow[rr, bend left, "pr_G|_K \text{ local iso}"] & G \times H \arrow[r, "pr_G"] & G \end{tikzcd}$$

$pr_G|_K : K \to G$ Lie gp-homo

$d_e(pr_G|_K) : \mathrm{graph}(\pi) \xrightarrow{\cong} \mathfrak{g}$ Lie alg. isom.

$\quad \shortparallel$

$d_e(pr_{\mathfrak{g}})|_{\mathrm{graph}(\pi)} \xRightarrow{\text{lemma}} pr_G|_K$ is a local isom $\Rightarrow \exists\, e_K \in W \subset K$ and $e_G \in V \subset G$ s.t.

$pr_G|_W : W \to V$ is an isom.

Consider $(pr_G|_W)^{-1} : V \to W$

$d_e(pr_G|_W)^{-1} : \mathfrak{g} \longrightarrow \mathrm{graph}(\pi)$

$X \longmapsto (X, \pi(X))$

Consider now $pr_H : G \times H \to H$ and $d_e pr_H : \mathfrak{g} \times \mathfrak{h} \to \mathfrak{h}$ Lie alg. homo. Then

$pr_H \circ (pr_G|_W)^{-1} : V \longrightarrow H$ is a

Lie gp local homo s.t.

$d_e\left(pr_H \circ (pr_G|_W)^{-1}\right)(X) =$

$= d_{e_G \times e_H} pr_H \; d_{e_G}(pr_G|_W)^{-1}(X) =$

$= d_{e_G \times e_H} pr_H \left(X, \pi(X)\right) = \pi(X)$ ∎

Thm (Ado) Any finite dim. Lie alg. $\mathfrak{g}$ is isom. to a subalgebra of $\mathfrak{gl}(n,\mathbb{R})$ for some n.

Corollary Any Lie gp G is locally isom. to a sbgp. of $GL(n,\mathbb{R})$ for some n.

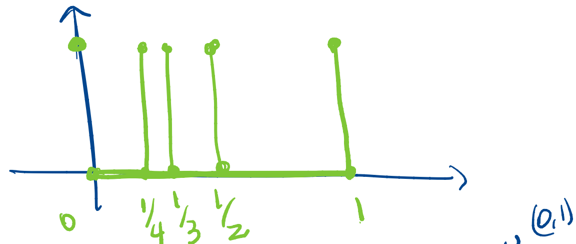
Recall If $G \le GL(n,\mathbb{R})$ the bracket in $\mathfrak{g}$ is the bracket in $\mathfrak{gl}(n,\mathbb{R})$.

In general Given a local homo when can we promote it to a global one?

Thm If G is a simply conn. topological gp, any local homo extends to a homo.

Recall • X is simply conn. if it is path-conn. & $\pi_1(X) = 0$.

• Path conn. ⇒ connected but the converse is not true



$X = \left\{\{\tfrac{1}{n}\} \times [0,1] : n \in \mathbb{N}\right\} \cup \left([0,1] \times \{0\}\right) \cup (0,1)$

However, conn + loc. path conn. ⇒ path. conn. (for ex. for Lie gps.)

Corollary (1) If G is a conn. Lie gp with Lie algebra $\mathfrak{g}$, ∃ simply conn. Lie gp $\tilde{G}$ with $\mathrm{Lie}(\tilde{G}) \cong \mathfrak{g}$.

(2) If two s.c. Lie gps have isom. Lie algebras ⇒ they are isomorphic.

(3) Given $\mathfrak{g}_1 \cong \mathfrak{g}_2$ ∃ s.c. Lie gps $\tilde{G}_i = \mathrm{Lie}(\mathfrak{g}_i)$.

This means that ∃ 1-1 corresp. between eq. classes of isom. Lie alg. and isom. classes of s.c. Lie gps.

Pf (1) Covering theory ⇒ if G is a conn. Lie gp, H is a top. gp, $p: H \to G$ covering map, there exists a (unique) Lie gp structure on H s.t. p is a Lie gp. homo and $\ker p$ is discrete.

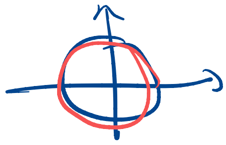
(2) $\mathfrak{g}_i = \mathrm{Lie}(\tilde{G}_i)$, $i = 1,2$.

$\mathfrak{g}_1 \cong \mathfrak{g}_2 \Rightarrow \exists\ p: U \to G_2$ local isom.; since $\tilde{G}_1$ is simply connected ⇒ p extends to a global homo $\tilde{p}: \tilde{G}_1 \to \tilde{G}_2$. This is a covering map & $\tilde{G}_2$ is simply connected ⇒ $\tilde{G}_1 \cong \tilde{G}_2$.

(3) Let $\mathfrak{g}_1 \cong \mathfrak{g}_2$. By Ado's thm ∃ G_i Lie gps.

(locally isom. to a sbgp of $GL(n,\mathbb{R})$) with $Lie(G_i)=\mathfrak{g}_i$, $i=1,2$. Let $\tilde{G}_i$ be the univ. covering. By (2), $\tilde{G}_1 \cong \tilde{G}_2$. ∎

Ex. $\varphi: \mathbb{R} \to S^1$
$t \mapsto e^{it}$



$\mathbb{R} = Lie(\mathbb{R}) \xrightarrow{\sim} Lie(S^1) = \mathbb{R}$

$Lie(S^1) \xrightarrow{\sim} Lie(\mathbb{R})$

$\psi : S^1 \to \mathbb{R}$

Corollary G Abelian $\iff$ $\mathfrak{g}$ is Abelian.

Recall $\mathfrak{g}$ is Abelian if $[X,Y]=0 \quad \forall\, X,Y \in \mathfrak{g}$.

$\left(\mathfrak{g} \subset \mathfrak{gl}(n,\mathbb{R}),\ [X,Y]=XY-YX \right)$

Pf ($\implies$) G Abelian $\iff$ $Inv: G \to G$ is a homo.

$\boxed{\text{Claim: } d_e Inv = -Id}$

If so,

$-[X,Y] = d_e Inv[X,Y] =$

$= [d_e Inv\, X, d_e Inv\, Y]$

$= [-X,-Y] = [X,Y] \implies$

$[X,Y]=0 \implies \mathfrak{g}$ Abelian

Pf of claim: $\varphi: (-\varepsilon,\varepsilon) \to G$ path s.t. $\varphi(0)=e$, $\psi(t) := Inv\, \varphi(t)$.

$\implies e = \varphi(t)\psi(t) \implies$

$\implies 0 = \frac{d}{dt}\Big|_{t=0} \varphi(t)\psi(t) =$

$= \varphi'(0)\psi(0) + \varphi(0)\psi'(0)$

$= \varphi'(0) + \psi'(0)$

If $\varphi'(0) = X \implies$

$\implies 0 = X + d_e Inv(X) \implies$

$\implies d_e Inv(X) = -X$. ∎

($\impliedby$) $\mathfrak{g}$ Abelian $\implies \mathfrak{g} \cong \mathbb{R}^n$ for some $n = \dim \mathfrak{g} \implies$

$\implies G$ is locally isom. to $\mathbb{R}^n$

$\left(\mathfrak{g} \xrightarrow{\sim} \mathbb{R}^n \implies \underset{G}{U \subset} \to \mathbb{R}^n \right)$

But $G \overset{(*)}{=} \bigcup_{n=1}^{\infty} U^n$ hence G is Abelian because it is locally Abelian.

(*) $\bigcup_{1}^{\infty} U^n$ is a sbgp & it is open $\overset{?}{\implies}$ closed. Since G is conn. $\implies G = \bigcup_1^\infty U^n$. ∎

Corollary (1) Any connected Abelian Lie gp G is is isom. to $\mathbb{R}^k \times \mathbb{T}^l$, $k,l \geq 0$.

(2) Any compact connected Abelian Lie gp is isom. to $\mathbb{T}^l$, $l \geq 0$.

(3) Any simply connected Abelian Lie gp is isom. to $\mathbb{R}^k$, $k \geq 0$.

Why $\implies$?

$H \leq G$ sbgp of a top. gp. that is open $\implies H$ is closed.

$G \setminus H = \bigcup_{\substack{g \in G \\ g \neq e}} gH$

Proof G Abelian $\Rightarrow$ $\mathfrak{g}$ Abelian $\Rightarrow$
$\Rightarrow$ $\mathfrak{g} \cong \mathbb{R}^n$ for $n = \dim \mathfrak{g}$.
$\Rightarrow$ $\exists\, 0 \in U_0 \subset \mathbb{R}^n$ s.t.
$\varphi: U_0 \to G$ is a local isomorphism. $\mathbb{R}^n$ is s.c. $\Rightarrow$
$\Rightarrow$ φ can be extended to a homomorphism $\varphi: \mathbb{R}^n \to G$.

Claim $\ker\varphi$ is discrete.

In fact $d_e\varphi$ is an isom. $\Rightarrow$
$\Rightarrow$ $\exists\, e \in U_e \subset G$ s.t.
$\varphi: U_0 \to \varphi(U_0) \cap U_e$
is a diffeo. $\Rightarrow$ $\ker(\varphi) \cap U_0 = \{0\}$ $\Rightarrow$ $\{0\}$ is open in $\ker(\varphi)$ and closed $\Rightarrow$

$\Rightarrow$ $\ker(\varphi)$ is discrete.

Exercise $D \leq \mathbb{R}^n$ discrete.
Then $\exists\, x_1, \dots, x_k \in D$ s.t.
(1) $D = \mathbb{Z}$-span of $\{x_1, \dots, x_n\}$,
$D = \mathbb{Z}x_1 \oplus \dots \oplus \mathbb{Z}x_n$, and
(2) $x_1, \dots, x_k$ are linearly indep. over $\mathbb{R}$

$\Rightarrow$ $\exists\, x_1, \dots, x_k \in \ker\varphi$, l.i. over $\mathbb{R}$. Let us write
$\mathbb{R}^n = V \oplus W$, $\dim W = n-k$
and $V = \mathbb{R}$-span$\{x_1, \dots, x_k\}$
The homo $\varphi: V \oplus W \to G$ is surjective because
$$G = \bigcup_{n=1}^{\infty} (\varphi(U_0) \cap U_e)^n \Rightarrow$$

$$\Rightarrow G \cong V \oplus W / \ker\varphi =$$
$$= V/\ker\varphi \oplus W$$
$$= (\mathbb{R}/\mathbb{Z})^k \oplus \mathbb{R}^{n-k}$$
$$= \mathbb{T}^k \oplus \mathbb{R}^\ell$$

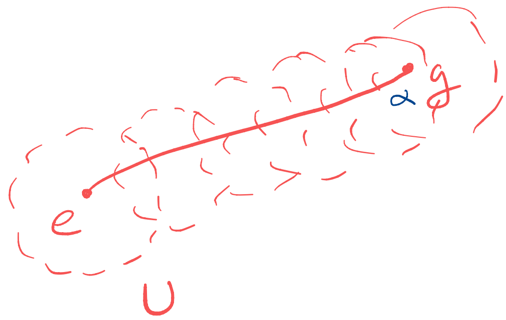
Now to the proof that a local homo of a s.c. gp can be extended to a global homo.

Sketch of the proof

Let $U \subset G$ be a nbd of $e \in G$ and $\varphi: U \to G$ a

local homo. Three steps:

(1) Use that G is path-connected to define φ on G.



(2) Use that $\pi_1(G) = 0$ to show that $\varphi_\alpha(g)$ is independent of α.

(3) Show that φ is a continuous homo.

(4) Show that φ is unique.

$\exp: \mathfrak{g} \to G$