Exercise Class: Lie Groups, Nov 5^{th}

Recap:

- Every Lie group G comes with its Lie algebra of left-invariant vector fields

$$\mathfrak{g} = \mathrm{Lie}(G) = \mathrm{Vect}(G)^G$$

- There is a vector space isomorphism

$$\begin{aligned} T_eG &\xrightarrow{\sim} \mathfrak{g} = \mathrm{Vect}(G)^G \\ v &\longmapsto (g \mapsto d_eL_g(v)) \\ X_e &\longleftarrow X \end{aligned}$$

- Important example:
$G = GL_n(\mathbb{R})$, $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{R}) \cong \mathbb{R}^{n\times n}$.

Via the above identification the Lie bracket is given by the commutator of matrices:

$$[A,B] = AB - BA \qquad \forall A,B \in \mathfrak{gl}_n(\mathbb{R}).$$

- Any Lie group hom. $\varphi: H \to G$ induces a Lie algebra homomorphism $d\varphi: \mathfrak{h} \to \mathfrak{g}$ via its differential

$$\begin{tikzcd} \mathfrak{h} \arrow[r, dotted, "d\varphi"] & \mathfrak{g} \\ T_eH \arrow[u, "\wr"] \arrow[r, "d_e\varphi"] \arrow[ur, phantom, "="] & T_eG \arrow[u, "\wr"'] \end{tikzcd}$$

What about the converse statement?

H, G Lie groups with Lie alg. $\mathfrak{h}, \mathfrak{g}$ and $\psi: \mathfrak{h} \to \mathfrak{g}$ Lie alg hom.

<u>Q:</u> Is there a Lie gp. hom. $\varphi: H \to G$ such that $d\varphi = \psi$?

<u>A:</u> In general: No!

<u>Counterexample:</u> $\mathrm{Lie}(\mathbb{R}) \cong \mathbb{R} \cong \mathrm{Lie}(S^1)$

But $\nexists\ \varphi: S^1 \to \mathbb{R}$ Lie gp. hom that induces $\mathrm{Lie}(S^1) \cong \mathrm{Lie}(\mathbb{R})$.

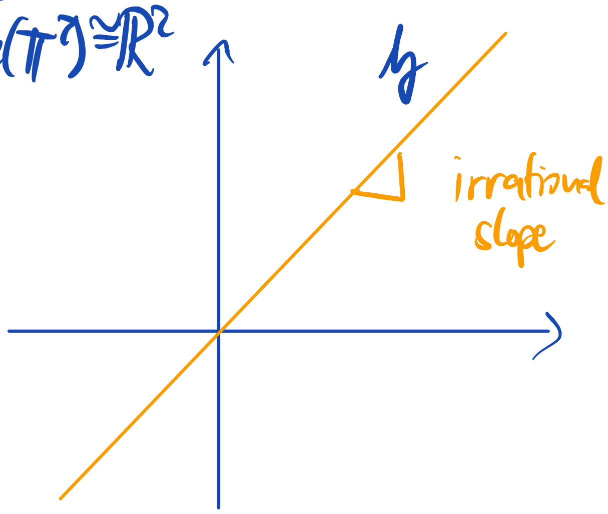
However, under some conditions the answer is: Yes!

- <u>First version of a converse:</u> (Starting with a subalgebra)

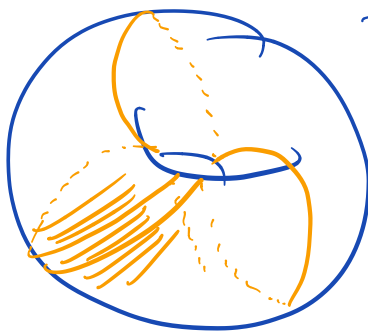
<u>Thm:</u> G Lie gp, $\mathfrak{g} = \mathrm{Lie}(G)$, $\mathfrak{h} \leq \mathfrak{g}$ a Lie subalg.
Then there exists an <u>immersed</u> Lie subgroup $H \subseteq G$ with Lie algebra $\mathfrak{h}$.

<u>Ex:</u> $G = \mathbb{T}^2 = S^1 \times S^1 \cong \mathbb{R}^2/\mathbb{Z}^2$

$\mathrm{Lie}(\mathbb{T}^2) \cong \mathbb{R}^2$



$\mathbb{T}^2 = G$



$\mathbb{R} \subset \mathbb{T}^2$ is <u>dense</u>

- <u>Ado's thm:</u>
Every (abstract) Lie algebra $\mathfrak{g}$ is a subalgebra of $\mathfrak{gl}_n(\mathbb{R})$ for some n.

↳ For every Lie algebra $\mathfrak{g}$ there is an immersed Lie subgroup $G \leq GL_n(\mathbb{R})$ with $\mathrm{Lie}(G) \cong \mathfrak{g}$.

- <u>Second version of a converse</u>

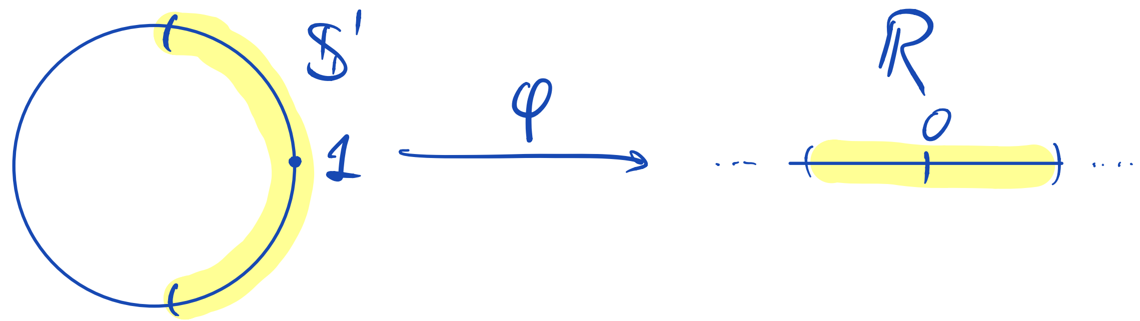
<u>Thm:</u> Let H, G be Lie groups and $\psi: \mathfrak{h} \to \mathfrak{g}$ a Lie algebra homomorphism. Then there is a <u>local</u> Lie group hom.

$\left[g, h \in U \ \& \ g \cdot h \in U \Rightarrow \varphi(g h) = \varphi(g)\varphi(h) \right]$

$\varphi : U \to G$ such that $d_e\varphi = \psi$.

open nbhd of $e \in H$

<u>Ex:</u> $H = S^1$, $G = \mathbb{R}$, $\mathrm{Lie}(S^1) = \mathrm{Lie}(\mathbb{R})$



<u>Q:</u> When can we extend φ to a <u>global</u> homomorphism?

<u>A:</u> If H is simply connected!

(Moreover, this homomorphism is uniquely determined by its "differential" $d\varphi: \mathfrak{h} \to \mathfrak{g}$)

Summarizing: We obtain an equivalence of categories!

$\mathrm{LieGp}_{s.c.}$: Category of SIMPLY CONNECTED Lie groups with (global) homomorphisms

LieAlg: Category of Lie algebras with Lie algebra homomorphisms

EQUIVALENCE

$$\mathrm{LieGp}_{sc} \longleftrightarrow \mathrm{LieAlg}$$

$$G \longmapsto \mathfrak{g}$$

$$(\varphi : G \to H) \longmapsto (d\varphi : \mathfrak{g} \to \mathfrak{h})$$

$$\tilde{G} \longleftarrow\!\!\shortmid \ \mathfrak{g}$$

universal cover of G, where $G \subseteq GL_n(\mathbb{R})$ s.t. $\mathrm{Lie}(G) = \mathfrak{g}$.

$$(\varphi : \tilde{G} \to \tilde{H}) \longleftarrow\!\!\shortmid \ (\psi : \mathfrak{g} \to \mathfrak{h})$$

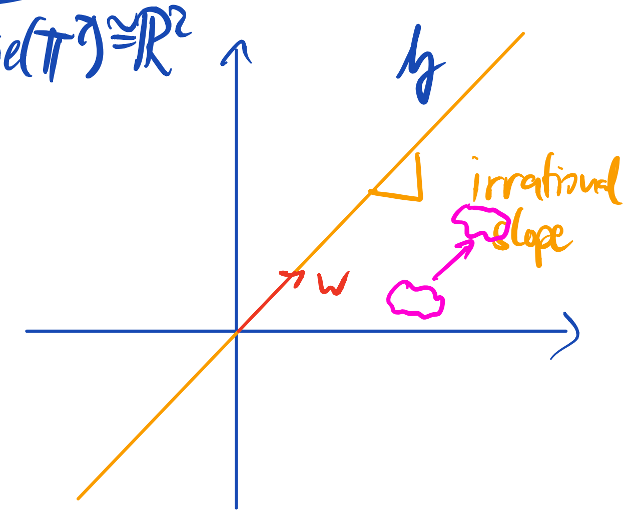
with $d\varphi = \psi$.

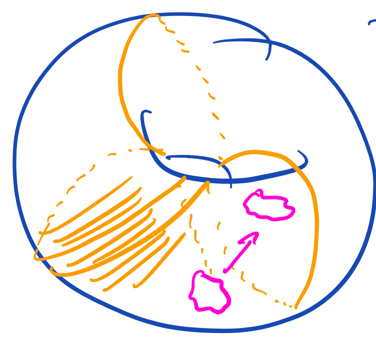
BREAK

Back to the previous example:

<u>Ex:</u> $G = \mathbb{T}^2 = S^1 \times S^1 \cong \mathbb{R}^2/\mathbb{Z}^2$

$\mathrm{Lie}(\mathbb{T}^2) \cong \mathbb{R}^2$





$\mathbb{T}^2 = G$

$\mathbb{R} \subset \mathbb{T}^2$ is <u>dense</u>

Let's use the notation $\pi : \mathbb{R}^2 \longrightarrow \mathbb{T}^2 = S^1 \times S^1$

$$x = (x_1, x_2) \longmapsto \left(\exp(2\pi i x_1), \exp(2\pi i x_2)\right)$$

Observe that $\mathbb{R}$ acts on $\mathbb{T}^2$ via

$$t * \pi(x) := \pi(x + t \cdot w)$$

Then the immersed copy of $\mathbb{R}$ in $\mathbb{T}^2$ is the orbit

$$\mathbb{R} * (1,1) \subset \mathbb{T}^2$$

<u>Q:</u> Can we say something quantitative about the <u>density</u> of orbits?

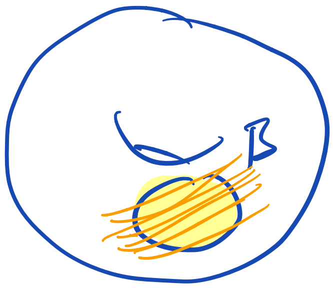
<u>A:</u> Yes! Let μ be the Haar probability measure on $\mathbb{T}^2$.
We want to show that for every $f \in L^2(\mathbb{T}^2, \mu)$

$$\frac{1}{T}\int_0^T f(t * x)\, dt \xrightarrow[(T \to \infty)]{\text{in } L^2(\mathbb{T}^2, \mu)} \int_{\mathbb{T}^2} f \, d\mu \qquad (*)$$

"time average" "space average"

Actually, one can show that the convergence in (*) is pointwise for almost every $x \in \mathbb{T}^2$ (Birkhoff's ergodic theorem)

<u>Interpretation:</u> Plug-in $f = \mathbb{1}_B$ = characteristic fct of some Borel set $B \subset \mathbb{T}^2$



Then $\int_0^T \mathbb{1}_B(t * x)\,dt$ is how much "time" the orbit $\mathbb{R} * x$ spent in B (up to time T).

Thus (*) means that after a long time the proportion of time that the orbit $\mathbb{R} * x$ has spent in B is (almost) equal to the proportion of space that B takes up in $\mathbb{T}^2$.

<u>Proof of (*):</u> Let $f \in L^2(\mathbb{T}^2, \mu)$.

Hilbert space with basis $\{e_v\}_{v \in \mathbb{Z}^2}$ where $e_v(\pi(x)) = \exp(2\pi i \langle v, x\rangle)$

We define $f_T(x) := \frac{1}{T}\int_0^T f(t * x)\,dt \in L^2(\mathbb{T}^2,\mu)$

We compute:

$$\langle f_T, e_v\rangle_{L^2} = \int_{\mathbb{T}^2} f_T(x)\cdot\overline{e_v(x)}\,d\mu(x)$$

$$= \int_{\mathbb{T}^2} \frac{1}{T}\int_0^T f(t * x)\,dt \cdot \overline{e_v}(x)\,d\mu(x)$$

$$(\mu \text{ is } \mathbb{R}\text{-inv.}) = \int_{\mathbb{T}^2} \frac{1}{T}\int_0^T f(x)\,\overline{e_v((-t) * x)}\,dt\,d\mu(x)$$

$$= \int_{\mathbb{T}^2} \frac{1}{T}\int_0^T f(x)\,\underbrace{\overline{\exp(2\pi i\langle v, x - t\cdot w\rangle)}}_{=\exp(-2\pi i\langle v,x\rangle)\cdot\exp(2\pi i\langle v,w\rangle t)}\,dt\,d\mu(x)$$

$$= \int_{\mathbb{T}^2} f(x)\exp(-2\pi i\langle v,x\rangle)\,d\mu(x)\cdot$$

$$\cdot\frac{1}{T}\int_0^T \exp(2\pi i\langle v,w\rangle t)\,dt$$

$$= \langle f, e_v\rangle_{L^2}\cdot\underbrace{\frac{1}{T}\int_0^T \exp(2\pi i\langle v,w\rangle t)\,dt}_{=:\,I_T(v)}$$

1st case: $v = 0 \in \mathbb{Z}^2$: $I_T(v) = 1$

$$\langle f_T, e_v\rangle_{L^2} = \langle f, e_0\rangle_{L^2} = \int_{\mathbb{T}^2} f\,d\mu$$

2^{nd} case: $v \neq 0$: Because w has irrational slope $\langle v, w\rangle \neq 0$.

$$|I_T(v)| \overset{s = \langle v,w\rangle t}{=} \frac{1}{T\cdot|\langle v,w\rangle|}\left|\int_0^{T\cdot\langle v,w\rangle} \exp(2\pi i s)\, ds\right|$$

"floor"

$$\leq \frac{1}{T\cdot|\langle v,w\rangle|}\left|\cancel{\int_0^{\lfloor T\cdot\langle v,w\rangle\rfloor} \exp(2\pi i s)\, ds}\right| \quad 0$$

$$+ \frac{1}{T\cdot|\langle v,w\rangle|}\underbrace{\int_{\lfloor T\cdot\langle v,w\rangle\rfloor}^{T\cdot\langle v,w\rangle} \overbrace{|\exp(2\pi i s)|}^{=1}\, ds}_{\leq |\langle v,w\rangle|} \xrightarrow[(T\to\infty)]{} 0$$

$$\Rightarrow \langle f_T, e_v\rangle_{L^2} \longrightarrow 0 \quad (T\to\infty)$$

$$\Rightarrow f_T \xrightarrow[(T\to\infty)]{(\text{in } L^2)} \langle f, e_0\rangle \cdot e_0 = \int_{\mathbb{T}^2} f\, d\mu \qquad \square$$

Statements like these are part of an area of mathematics called "Dynamical Systems".

(It will not play a role in this course, but it's good to know for "culture".)