11 November 2020

Theorem If G is simply conn. top. gp and $\varphi: U \to H$ (top. gp) is a local homo, then φ extends uniquely to a homo $\varphi: G \to H$.

Recall simply conn = path conn. & $\pi_1(G) = 0$.

Pf (1) Use path-conn. to define φ on G

(2) Use $\pi_1(G)=0$ to show that defn. is indep. of path.

(3) φ continuous, homo & !

Let $g \in G$ and let $\alpha: [0,1] \to G$ be a cont. path $\alpha(0) = e$, $\alpha(1) = g$.

Choose a partition of α $\{t_k\}_{k=0}^{n}$ s.t.

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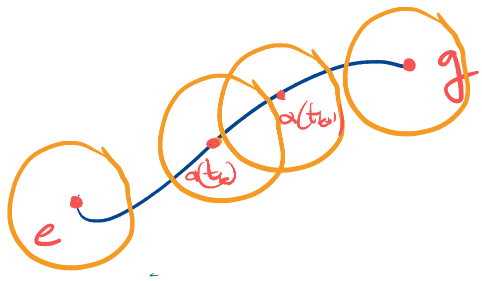

$t_0 = 0$, $t_n = 1$, if $I_k = [t_{k-1}, t_k]$, $\forall s,t \in I_k$

$\Rightarrow \alpha(s)^{-1}\alpha(t) \in U$.

Call such a partition good.

Choose $W \ni e$ s.t. $W = W^{-1}$, $W^2 = WW^{-1} \subset U$ and $\alpha \subset \bigcup_{k=0}^{n} \alpha(t_k) W$

useful later

Such a partition exists because

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$\exists \delta > 0$ s.t. $\alpha(s)^{-1}\alpha(t) \in U$ $\forall$ $s,t \in I$ with $|s-t| < \delta$.

Set $x_k := \alpha(t_k)$, $x_0 = e$, $x_n = g$.

$g = (x_0^{-1}x_1)(x_1^{-1}x_2)\dots(x_{n-1}^{-1}x_n)$

with $x_j^{-1}x_{j+1} \in U$ $\forall$ $j = 0, \dots, n-1$

Define $\varphi_\alpha(g) := \varphi(x_0^{-1}x_1)\dots\varphi(x_{n-1}^{-1}x_n)$.

Show that φ_α is indep. of the partition.

Let $t \in I_k = [t_{k-1}, t_k] =$

$= [t_{k-1}, t] \cup [t, t_k]$

$\Rightarrow t \in I_k \Rightarrow$ (1) $\alpha(t_{k-1})^{-1}\alpha(t) \in U$,

(2) $\alpha(t)^{-1}\alpha(t_k) \in U$

(3) $\alpha(t_{k-1})^{-1}\alpha(t_k) \in U$

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$\Rightarrow \varphi(x_{k-1}^{-1}x_k) = \varphi(x_{k-1}^{-1}x)\varphi(x^{-1}x_k)$ where $x = \alpha(t)$.

$\Rightarrow \varphi_\alpha$ does not depend on the partition.

Use: Any refinement of a good partition is a good partition.

(2) $\pi_1(G) = 0 \Rightarrow$ choose a homotopy $H: [0,1] \times [0,1] \to G$ with $H(s,0) = \alpha_0(s)$

$H(s,1) = \alpha_1(s)$

and we set $\varphi_t(g) := \varphi_{\alpha_t}(g)$

Let $\delta > 0$ be such that

$H(s_1, t_1)^{-1} H(s_2, t_2) \in W$

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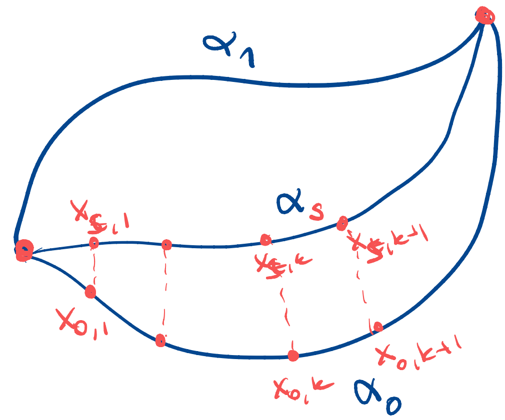
$\forall\, s_1, s_2, t_1, t_2 \in [0,1]$ s.t. $|s_1 - s_2| + |t_1 - t_2| < \delta$.

Then for all $s \in [0,1]$ the partition $\{t_k\}_{k=0}^{n} = \left\{\frac{k}{n}\right\}_{k=0}^{n}$ is good, provided we choose n large enough that $\frac{1}{n} < \frac{\delta}{2}$.



$A := \{ s \in [0,1] : \varphi_s(g) = \varphi_0(g) \}$

To show that A is open & closed ($A \neq \emptyset$, $0 \in A$).

<u>A closed</u> $(s_j)_{j \in \mathbb{N}}$, $s_j \to s$, $s_j \in A$. Want to show $s \in A$.

Let α_{s_j}, α_s, $\{t_k\}$ good partition. Continuity of H

$$\Rightarrow \lim_{j \to \infty} \alpha_{s_j}(t_k) = \alpha_s(t_k)$$

Write
$$x_{s_j,k} = \alpha_{s_j}(t_k)$$
$$x_{s,k} = \alpha_s(t_k)$$

$$\lim_{j \to \infty} \varphi(x^{-1}_{s_j,k}\, x_{s_j,k+1}) = \varphi(x^{-1}_{s,k}\, x_{s,k+1})$$

$\Rightarrow$

$$\varphi_{s_j}(g) = \prod_{0}^{n-1} \varphi(x^{-1}_{s_j,k}\, x_{s_j,k+1}) \longrightarrow$$

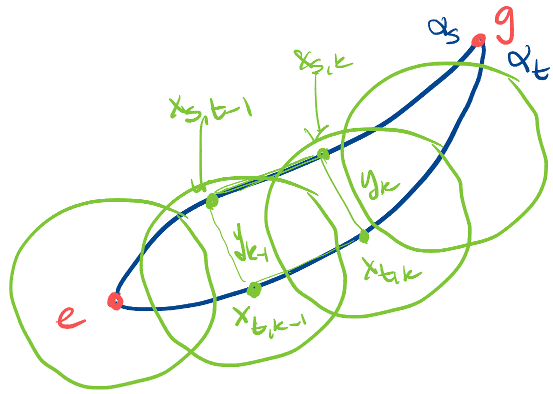
$$\longrightarrow \prod_{0}^{n-1} \varphi(x^{-1}_{s,k}\, x_{s,k+1}) =$$

$$= \varphi_s(g)$$

constantly equal to $\varphi_0(g)$ $\forall j$

$\Rightarrow$ $\varphi_s(g) = \varphi_0(g) \Rightarrow s \in A$.

<u>A open</u> let $t \in A$ and let $s \in A$ be close enough that $\alpha_s \subset \bigcup_{j=0}^{n} x_j W$.



$$x^{-1}_{s,k-1}\, x_{s,k} = y_{k-1}\, x^{-1}_{t,k-1}\, x_{t,k}\, y_k^{-1}$$

$$\Rightarrow \varphi_s(g) = \prod_{k=1}^{n} \varphi(x^{-1}_{s,k-1}\, x_{s,k}) =$$

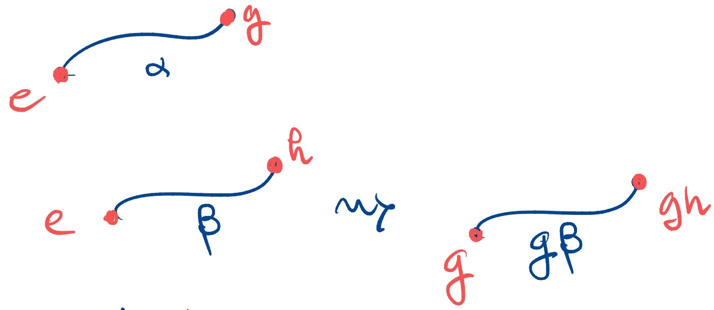
$$= \prod_{1}^{n} \varphi(y_{k-1})\, \varphi(x^{-1}_{t,k-1}\, x_{t,k})\, \varphi(y_k^{-1})$$

φ local homo and these pts behave "well".

This is a telescopic product & $\varphi(y_0) = e = \varphi(y_n)$

(3) • Continuity it is easy

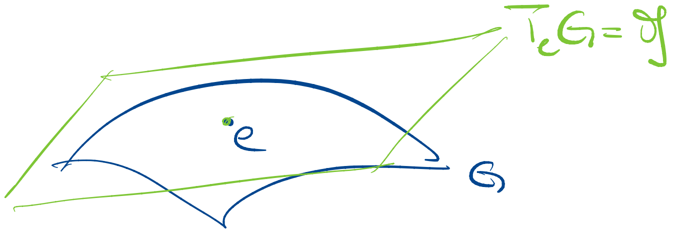
• φ is a homo:



and by defn.

$$\varphi(gh) = \varphi(g)\varphi(h).$$

• φ is unique because φ is defn. from its values on U and $G = \bigcup_{n=1}^{\infty} U^n$. ∎



Defn. Let G be a Lie gp. A one-parameter subgroup of G is a Lie gp. homo

$$\varphi : \mathbb{R} \longrightarrow G.$$

G Lie group, $\mathrm{Lie}(G) = \mathfrak{g}$, $X \in \mathfrak{g}$,

$$\mathrm{Lie}(\mathbb{R}) \longrightarrow \mathfrak{g}$$
$$t \longmapsto tX$$

$\Rightarrow \exists$ local homo φ that can be extended to a global homo because $\mathbb{R}$

is simply connected.

Call it $\varphi_X : \mathbb{R} \to G$ and observe that $d_0\varphi_X(t) = tX$.

Definition The exponential map of G (of $\mathfrak{g}$) is defn by

$$\exp_G : \mathfrak{g} \longrightarrow G$$
$$X \longmapsto \varphi_X(1)$$

Proposition $G, \mathfrak{g} \ni X$. Let $\tilde{X} \in \mathrm{Vect}(G)^G$ with $\tilde{X}_e = X$.

1. φ_X is an integral curve of $\tilde{X}$ and the unique one such that $\varphi_X(0) = e$. Also $L_g \circ \varphi_X$ is the only

integral curve through g at $t = 0$. In particular left inv. vector fields are complete (i.e. integral curves are defined $\forall t \in \mathbb{R}$).

2. $\exp(tX) = \varphi_X(t) \Rightarrow$ $t \mapsto \exp(tX)$ is the unique 1-par. sbgp. corresponding to X, that $d_0(\exp tX) = d_0\varphi_X(t) = X$.

3. $\exp((t_1+t_2)X) = (\exp t_1X)\cdot(\exp t_2X)$ $\forall\, t_1, t_2 \in \mathbb{R}$

4. $\exp(tX)^{-1} = \exp(-tX)$ $\forall\, t \in \mathbb{R}$

5. $\exp: \mathfrak{g} \to G$ is a smooth map and a local diffeo from a nbd of $0 \in \mathfrak{g}$ to a nbd of $e \in G$.

Pf (1) $\tilde{1} \in Vect(\mathbb{R})^{\mathbb{R}}$, $\tilde{X} \in Vect(G)^{G}$ are φ_X-related

Lemma G, H Lie gps, $\mathfrak{g}, \mathfrak{h}$, $\varphi: G \to H$ gp. homo. Then the left inv. v.f. defn. by $X \in \mathfrak{g}$, $d_e\varphi(X) \in \mathfrak{h}$ are φ-related.

$\Rightarrow$ φ_X is the unique int. curve of $\tilde{X}$ s.t. $\varphi_X(0) = e$. $L_g \varphi_X$ is the unique int. curve of $\tilde{X}$

through g at $t=0$, because $\tilde{X}$ is left-invariant.

(2) We claim that

$$\boxed{\varphi_{sX}(t) = \varphi_X(st)}, \quad \forall s,t \in \mathbb{R}$$

If so, by taking $t=1$ we get

$$\varphi_{sX}(1) \underset{\text{defn.}}{=} \varphi_X(s)$$

$\Rightarrow$ $s \longmapsto \exp(sX) = \varphi_X(s)$ is the unique 1-par. sbgp. whose tg vector at $t=0$ is X.

Now we show $\boxed{}$.

φ_{sX} is the int. curve of sX through e at $t=0$

$\eta: \mathbb{R} \to G$

$\eta(t) = \varphi_X(st)$ for $s \in \mathbb{R}$ fixed.

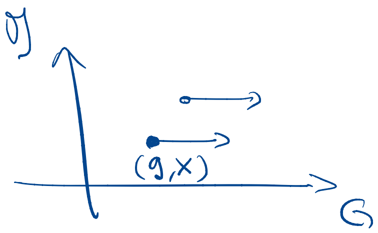
$\eta(0) = \varphi_X(0) = e$

$$d_0\eta(t) = \frac{d}{dt}\Big|_{t=0} \varphi_X(st) = s\,\varphi_X'(0) = sX$$

$\Rightarrow$ $\varphi_X(st) = \varphi_{sX}(t)$.

(3) & (4) follow since $t \longmapsto \exp(tX)$ is a homo.

(5) $M = G \times \mathfrak{g}$ & consider $\Xi \in Vect(G \times \mathfrak{g})$ a "horizontal" v.f.



$$\Xi(g,X) = (\tilde{X}_g, 0) \in T_gG \times T_X\mathfrak{g}$$

$\tilde{X}$ smooth $\Rightarrow$ Ξ smooth $\Rightarrow$ $\exists$ integral curve $\varphi_{(\tilde{X}_g,0)}: (-\varepsilon,\varepsilon) \to G \times \mathfrak{g}$ through (g,X) at $t=0$ given by

$$t \longmapsto (g\exp tX, X) = L_g(\exp tX, X)$$

But $G \times \mathfrak{g}$ is a Lie gp. $\Rightarrow$ $\varphi_{(\tilde{X}_g,0)}$ is defn. on $\mathbb{R}$

since Ξ is complete.
So we can consider

$$\varphi_{(\tilde{X}_g,0)}(1) = (g\exp X, X)$$

Thm. of smooth dep. of sol. of ODE w.r.t. initial conditions $\Rightarrow$ $\varphi_{(\tilde{X}_g,0)}$ is smooth on $G\times\mathfrak{g}$.

$pr_G : G\times\mathfrak{g} \longrightarrow G$

and

$$\exp(X) = pr_G \circ \varphi_{(X,0)}(1)$$

To show that exp is a local diffeo, enough to show that

$d_0\exp : T_0\mathfrak{g} \longrightarrow T_eG$

is non-singular.

We'll show it is the identity.

Let $\psi : (-\varepsilon,\varepsilon) \longrightarrow \mathfrak{g}$,

$\psi(t) = tX$

$\psi(0) = 0$, $\psi'(0) = X$.

But $\varphi_X : \mathbb{R} \to G$

$\varphi_X(t) = \exp(tX)$

$\varphi_X(0) = 0$, $\varphi_X'(0) = X$

$\Rightarrow d_0\varphi_X = \psi$.

$\Rightarrow d_0\varphi_X(t) = tX$

$d_0\exp(tX) = tX$

$\Rightarrow d_0\exp = \mathrm{Id}$ $\blacksquare$