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$\exp_G : \mathfrak{g} \to G$ $\exp_G(X) := \varphi_X(1)$,
where $\varphi_X : \mathbb{R} \to G$ is a 1-par. sbgp, which is the unique integral curve with $\varphi_X(0) = e$ and $\varphi_X'(0) = X$.

Properties

(1) $\exp(tX) = \varphi_X(t)$

(2) $\exp((t+s)X) = \exp(tX)\exp(sX)$

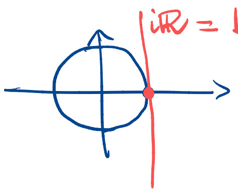
$\exp(tX)^{-1} = \exp(-tX)$

(3) exp is a smooth map and a local diffeo $\exp : U_0 \to U_e$

Example

(1) $G = \mathbb{R}^n$, $\exp : \mathrm{Lie}(\mathbb{R}^n) \to \mathbb{R}^n$ is the identity

(2) $G = S^1 = \{ z \in \mathbb{C} : |z| = 1 \}$



$i\mathbb{R} = \mathrm{Lie}(S^1)$ with $\exp : \mathbb{R} \simeq i\mathbb{R} \to S^1$
$t \longmapsto e^{it}$

(3) next: the $\exp_{GL(n,\mathbb{C})}$ = matrix exponential.

Lemma Let V be a complex vector space.

(1) The map

$$\mathrm{End}(V) \longrightarrow \mathrm{End}(V)$$
$$X \longmapsto e^X := \sum_{j=0}^{\infty} \frac{X^j}{j}$$

is well defined and takes values into $GL(V) < \mathrm{End}(V)$

(2) $\det(e^X) = e^{\mathrm{tr}\, X}$ $\forall X \in \mathrm{End}(V)$

(3) If X, Y commute then

$$e^{X+Y} = e^X e^Y.$$

Corollary $t \longmapsto e^{tX}$ is a smooth curve in $GL(V)$ that takes value I at $t=0$ and whose tg vector at $t=0$ is X.

Thus $e^{tX} = \exp(tX)$ so
$\exp : \mathfrak{gl}(V) \longrightarrow GL(V)$
is $\exp(X) = e^X$.

Pf of corollary

By (3) in Lemma $\Rightarrow$
$\Rightarrow$ e^{tX} is a homomorphism, hence we need to compute

$$\frac{d}{dt}\Big|_{t=0} e^{tX} = \frac{d}{dt}\Big|_{t=0} \left(\sum_{j=0}^{\infty} \frac{(tX)^j}{j!} \right) =$$

$$= \sum_{j=1}^{\infty} \frac{t^{j-1} X^j}{j!} \Big|_{t=0} =$$

$$= X \sum_{j=1}^{\infty} \frac{t^{j-1} X^{j-1}}{j!} =$$

$$= X e^{tX} \Big|_{t=0} = X. \quad \blacksquare$$

Pf of lemma (1) Need to see that $\sum_{j=0}^{\infty} \frac{X^j}{j!}$ converges.

We'll see that it converges unif. on cpt sets. Let $K \subset \mathrm{End}(V)$ be a cpt set and let $C > 0$ be such that

$$|X_{ij}| \leq C \quad \forall X \in K.$$

Induction $|(X^m)_{ij}| \leq (nC)^m$

Since $\sum_{m=0}^{\infty} \frac{(nC)^m}{m!}$ converges by Weierstrass test $\Rightarrow$

$\Rightarrow$ $\sum_{m=0}^{\infty} \frac{X^m}{m!}$ converges uniformly on K.

Now we show that $e^X \in GL(V)$.

$$S_j(X) := \sum_{m=0}^{j} \frac{X^m}{m!}$$

Multipl. on End(V) is cont.

$\Rightarrow$ $X \longmapsto BX \quad \forall B \in \mathrm{End}(V)$

is cont. $\Rightarrow$

$$B \lim_j S_j(X) B^{-1} =$$

$$= \lim_j B S_j(X) B^{-1}$$

$$\Rightarrow \boxed{B e^X B^{-1} = e^{BXB^{-1}}}$$

We are over $\mathbb{C}$ because we want to say that $\exists$ $B \in GL(V)$ s.t. BXB^{-1} is

upper triangular. In fact

v_1 eigenvector of X

$\vdots$

v_{j+1} eigenvector of $pr_j \circ X$,

where $pr_j : V \to W_j$,

$V = V_j \oplus W_j$,

$V_j =$ span of $\{v_1, \dots, v_j\}$

$\dim W_j = \dim V - j$.

If λ_j is the eigenvalue corresp. to v_j $\Rightarrow$

$$BXB^{-1} = \begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \Rightarrow$$

$$e^{BXB^{-1}} = \begin{pmatrix} e^{\lambda_1} & & * \\ & \ddots & \\ 0 & & e^{\lambda_n} \end{pmatrix}$$

$$\Rightarrow \det e^{BXB^{-1}} \neq 0$$

$$\det e^X = \det(B e^X B^{-1}) =$$

$$= \det e^{BXB^{-1}} \neq 0. \Rightarrow$$

$\Rightarrow e^X \in GL(V)$.

(2) To show that

$$\det e^X = e^{\operatorname{tr} X}$$

$$\det e^X = \det(B e^X B^{-1}) =$$

$$= \det e^{BXB^{-1}} =$$

$$= e^{\lambda_1} \cdots e^{\lambda_n} = e^{\lambda_1 + \dots + \lambda_n} =$$

$$= e^{\operatorname{tr} X}.$$

(3) $$e^X e^Y = \left(\sum_{j=0}^{\infty} \frac{X^j}{j!}\right)\left(\sum_{m=0}^{\infty} \frac{Y^m}{m!}\right) =$$

$$\overset{(*)}{=} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{X^k Y^{n-k}}{k!(n-k)!}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \underbrace{\sum_{k=0}^{n} \binom{n}{k} X^k Y^{n-k}}_{(X+Y)^n} =$$

$$= \sum_{n=0}^{\infty} \frac{(X+Y)^n}{n!},$$

where we used in (*) that X, Y commute. ∎

Proposition (Naturality of exp)

$\varphi : G \to H$ Lie gp. homo. The following commutes

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "d_e\varphi"] \arrow[d, "\exp_G"'] \arrow[dr, phantom, "\mathrm{CD}"] & \mathfrak{h} \arrow[d, "\exp_H"] \\ G \arrow[r, "\varphi"] & H \end{tikzcd}$$

that is $\exp_H \circ d_{e_G}\varphi = \varphi \circ \exp_G$.

Pf   let $X \in \mathfrak{g}$. Then
$t \longmapsto \exp_G(tX)$ is the unique 1-par. subgp. of G which goes through e_G at $t=0$ with tg vector X.

Since φ is a homom., then $t \longmapsto \varphi(\exp_G(tX))$ is a 1-par. sbgp. of H that goes through e_H at $=0$ with tg vector

$$\frac{d}{dt}\Big|_{t=0} \varphi(\exp_G(tX)) =$$

$$= d_{e_G}\varphi \cdot \frac{d}{dt}\Big|_{t=0} \exp(tX)$$

$$= d_{e_G}\varphi(X)$$

$\Rightarrow$ The only 1-par. sbgp. of H with this property is $t \mapsto \exp_H(t\, d_e\varphi(X))$ $\Rightarrow$

$$\Rightarrow \varphi(\exp_G(tX)) =$$

$$= \exp_H(t\, d_e\varphi(X))$$

$$\Rightarrow \varphi \circ \exp_G = \exp_H \circ d_e\varphi \quad \square$$

$$\exp_G : \underset{\substack{\uparrow \\ \text{connected}}}{\mathfrak{g}} \to G \quad \Rightarrow$$

$\Rightarrow \exp(\mathfrak{g}) \subset G^0$

Question: Is exp onto?

Answer: Not (i.e. not necessarily).

Example $\exp_{SL(2,\mathbb{R})} : \mathfrak{sl}(2,\mathbb{R}) \to SL(2,\mathbb{R})$ is not surjective.

Proof Two steps:

(1) If $A \in \exp(\mathfrak{sl}(2,\mathbb{R}))$ then $\exists\, B \in \exp(\mathfrak{sl}(2,\mathbb{R}))$ s.t. $A = B^2$.

(2) We will show that there are elements in $SL(2,\mathbb{R})$ that are not squares.

Pf of (1) let $X \in \mathfrak{sl}(2,\mathbb{R})$,

$$A = \exp(X) = \exp\left(\frac{X}{2} + \frac{X}{2}\right) =$$

$$= \left(\exp\left(\frac{X}{2}\right)\right)^2 = B^2,$$

where $B = \exp\left(\frac{X}{2}\right)$.

Need to verify that $B \in \exp(\mathfrak{sl}(2,\mathbb{R}))$,

that is that $\frac{X}{2} \in \mathfrak{sl}(2,\mathbb{R})$.

But this is true since $\operatorname{tr} X = 0 \Rightarrow \operatorname{tr}\left(\frac{X}{2}\right) = 0$

Pf of (2) We will show that if $A \in SL(2,\mathbb{R})$ is a square, then $\operatorname{tr} A \geq -2$ and we will provide an ex. of a matrix with $\operatorname{tr} A < -2$.

For example $A = \begin{pmatrix} -2 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$

$\Rightarrow \operatorname{tr} A < -2$.

Any $B \in GL(2,\mathbb{R})$ is a root of its own charact. poly

$$p(\lambda) = \lambda^2 - \operatorname{tr}(B)\lambda + \det B$$

$$B^2 - \operatorname{tr}(B) B + (\det B) I = 0$$

If $B \in SL(2,\mathbb{R}) \Rightarrow$

$\Rightarrow B^2 - tr(B) B + 2 = 0$

$\Rightarrow tr(B^2) - (tr B)^2 + 2 = 0$

If $A = B^2 \Rightarrow$

$\Rightarrow tr A = (tr B)^2 - 2 \geq -2$. ∎

Question For which Lie gps is the exp surjective?

Ex. $N = \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \right\} < SL(n,\mathbb{R})$

$\mathfrak{n} = \left\{ \begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix} \right\} < \mathfrak{sl}(n,\mathbb{R})$

• If $X \in \mathfrak{n} \Rightarrow X^n = 0$

$X^2 = \begin{pmatrix} 0 & 0 & * \\ & \ddots & 0 \\ 0 & & 0 \end{pmatrix}$

$\Rightarrow \exp(X) = \sum_{j=0}^{\infty} \frac{X^j}{j!} =$

$= \sum_{j=0}^{n-1} \frac{X^j}{j!}$.

• If $A \in N \Rightarrow A = I + A'$, where now A' has the property that $(A')^n = 0$.

Define

$\log : N \to \mathfrak{n}$

$A \longmapsto \sum_{j=1}^{n-1} (-1)^{j-1} \frac{(A')^j}{j}$

Claim $\exp : \mathfrak{n} \to N$ and $\log : N \to \mathfrak{n}$ are inverse of each other

$\Rightarrow$ exp is jective

Theorem (Cartan) The exponential map of a compact connected Lie gp is surjective.

Remark In general exp is not surjective $\Rightarrow$

$\forall g \in G \ \exists \ X_1, \dots, X_n \in \mathfrak{g}$ s.t.

$g = \exp(X_1) \cdots \exp(X_n)$.

More recent, $n = 2$ suffices.

Idea of the proofs of Cartan's thm.

(1) One can see easily that $\exp : Lie(\mathbb{T}^n) \to \mathbb{T}^n$ is surjective. One should then know that in a compact connected Lie

gp every element lies in a maximal torus and all tori are conjugate.

(2) Uses Hopf-Rinow thm. for Riemannian manifolds.

M Riem. mfd compact & connected complete $\Rightarrow$

$\Rightarrow$ any two pts can be joined by a geodesic

$\Rightarrow$ Riem. exponential is surjective. One can prove that Riem. exp =

= Lie gp. exp $\Rightarrow$ onto.

Is a Lie gp. a Riem. mfd?

Yes if G is compact & connected because it can be given a bi-inv. metric.

Two ways to give G a bi-inv. metric.

(1) Peter-Weyl theorem $\Rightarrow$ embed G into $U(n)$, get the bi-inv. metric from $\mathbb{C}^{n\times n}$.

(2) Take a positive defn. inner product on $\mathfrak{g}$, and average it to make it invariant. we invariant pos. defn. inner

product on $\mathfrak{g}$ we smear it around TG to get the invariance on the other side.

Remark A Lie group G admits a bi-inv. metric iff it is the product of a compact group and an Abelian group.

Proposition Let G be a connected Abelian Lie gp. Then $\exp : \mathfrak{g} \to G$ is a group homo and

$$G \simeq \mathfrak{g}/\ker(\exp)$$

$\ker(\exp)$ is discrete.

Pf G Abelian $\Rightarrow$

$m : G\times G \to G$ homo

$m(g_1,h_1)\, m(g_2,h_2) =$

$= g_1h_1g_2h_2 = g_1g_2h_1h_2 =$

$= m(g_1g_2, h_1h_2)$

& $d_e m_{G\times G}(X,Y) = X+Y$.

$$\begin{tikzcd}
\mathfrak{g}\times\mathfrak{g} \arrow[r, "d_e m"] \arrow[d, "\exp_{G\times G}"'] \arrow[dr, phantom, "\circlearrowright"] & G \arrow[d, "\exp_G"] \\
G\times G \arrow[r, "m"] & G
\end{tikzcd}$$

$\Rightarrow$

$\exp_G(X+Y) = \exp_G \circ d_e m(X,Y) =$

$= m(\exp_{G\times G}(X,Y)) =$

$= m(\exp_G(X), \exp_G(Y)) =$

$= \exp_G(X)\exp_G(Y)$

$\Rightarrow \exp_G$ is a homom.

$\Rightarrow \exp_G : \mathfrak{g} \to G$

local diffeo $\exp_G : U_0 \to U_e$

$G = \bigcup_{n=0}^{\infty} U_e^n \Rightarrow$

$\Rightarrow \exp_G$ is onto