Exercise Class: Lie Groups, Nov. 19th

Exercise 1.(Related Vector Fields):

Let M, N be smooth manifolds and let $\varphi : M \to N$ be a smooth map. Recall that two vector fields $X \in \mathrm{Vect}(M)$, $X' \in \mathrm{Vect}(N)$ are called φ-related if

$$d_p\varphi(X_p) = X'_{\varphi(p)}$$

for every $p \in M$.

Show that $[X, Y]$ is φ-related to $[X', Y']$ if $X \in \mathrm{Vect}(M)$ is φ-related to $X' \in \mathrm{Vect}(N)$ and $Y \in \mathrm{Vect}(M)$ is φ-related to $Y' \in \mathrm{Vect}(N)$.

Solution:

Key insight: $X \overset{\varphi}{\sim} X'$ (φ-related)

$\forall f \in C^\infty(N)$ $\forall p \in M$

$$(X'f)(\varphi(p)) = X'_{\varphi(p)}.f = d\varphi_p(X_p).f$$
$$= X_p(f\circ\varphi),$$

i.e. $(X'f)\circ\varphi = X(f\circ\varphi)$

Now, compute:

$$[X', Y']_{\varphi(p)} f = X'_{\varphi(p)}(Y'f) - Y'_{\varphi(p)}(X'f)$$
$$= X_p((Y'f)\circ\varphi) - Y_p((X'f)\circ\varphi)$$
$$= X_p(Y(f\circ\varphi)) - Y_p(X(f\circ\varphi))$$

$$= d\varphi_p([X,Y]_p).f$$

$\square$

Exercise 3.(Some Lie Algebras):

a) Let M, N be smooth manifolds and let $f : M \to N$ be a smooth map of constant rank r. By the constant rank theorem we know that the level set $L = f^{-1}(q)$ is a regular submanifold of M of dimension $\dim M - r$ for every $q \in N$. Show that one may canonically identify

$$T_p L \cong \ker d_p f$$

for every $p \in L = f^{-1}(q)$.

Solution: Two steps: 1) $T_pL \subseteq \ker(d_pf)$

2) $\dim T_pL = \dim \ker(d_pf)$

1) Let $v \in T_pL$, and let $\gamma : (-\varepsilon, \varepsilon) \to L = f^{-1}(q)$ be a smooth curve with $\dot\gamma(0) = v$.

Then $d_pf(v) = \frac{d}{dt}\Big|_{t=0} \underbrace{f(\gamma(t))}_{= q \text{ (const.)}} = 0$

$\Rightarrow v \in \ker(d_pf)$.

2) $\dim \ker(d_pf) = \dim T_pM - \operatorname{rank}(d_pf)$

$= \dim M - r = \dim L = \dim(T_pL)$.

$\square$

b) We will compute the Lie algebra $\mathfrak{sp}(2n,\mathbb{C})$ of the Lie group $Sp(2n,\mathbb{C}) = \{g \in GL(2n,\mathbb{C}) \mid g^T J g = J\}$

where
$$J = \begin{array}{c} \\ n\{ \end{array}\begin{array}{c} \overbrace{}^{n} \\ \left(\begin{array}{c|c} 0 & I_n \\ \hline -I_n & 0 \end{array}\right) \end{array} :$$

Define
$$F: GL(2n,\mathbb{C}) \longrightarrow \mathbb{C}^{2n\times 2n}$$
$$g \longmapsto g^T J g$$

$$\Longrightarrow \quad Sp(2n,\mathbb{C}) = F^{-1}(J)$$

- <u>F has constant rank:</u>

$$d_g F(X) = \frac{d}{dt}\Big|_{t=0} F(g+t\cdot X)$$

$$= \frac{d}{dt}\Big|_{t=0} \left[(g+t\cdot X)^T \cdot J \cdot (g+t\cdot X)\right]$$

$$\overset{\text{Ex. 2}}{=} \underbrace{X^T J g}_{(g^T J^T X)^T = -(g^T J X)^T} + g^T J X$$

$$\overset{J^T = -J}{=} g^T J X - (g^T J X)^T$$

invertible

$$= P(g^T J X) = (P \circ L_{g^T J})(X)$$

where $P(Y) = Y - Y^T$.

$\Longrightarrow$ F has constant rank.

By (a): $T_I Sp(2n, \mathbb{C}) = \ker(d_I F)$

$= \{X \in \mathbb{C}^{2n \times 2n} \mid X^T J + J X = 0\}$

$\overset{(*)}{=} sp(2n, \mathbb{C})$

<u>Q</u>: What's the Lie bracket?

<u>A</u>: Because $Sp(2n, \mathbb{C}) \leq GL(2n, \mathbb{C})$, inclusion induces an injective Lie alg. hom

$$\begin{array}{ccc} d\iota : sp(2n, \mathbb{C}) & \hookrightarrow & gl(2n, \mathbb{C}) \\ \cong & & \cong \\ T_e Sp(2n, \mathbb{C}) & \subseteq & T_e GL(2n, \mathbb{C}) \end{array}$$

In $gl(2n, \mathbb{C})$ the Lie bracket is given by the commutator:

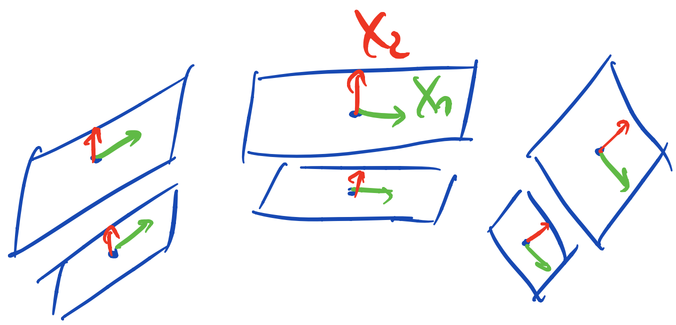
$[A, B] = AB - BA \quad \forall A, B \in gl(2n, \mathbb{C})$

Recap: Distributions on manifolds

A (smooth) <u>distribution</u> $\mathcal{D}$ on M (of dimension k) is a collection of tangent spaces $\mathcal{D} = \{D_p\}_{p\in M}$ where $D_p \subseteq T_pM$ $\forall p \in M$ such that for every point $p \in M$ there is an open nbhd. $U \ni p$ and linearly indep. smooth vector fields $X_1, \dots, X_k \in \mathrm{Vect}(U)$ that span

$$D_q = \langle (X_1)_q, \dots, (X_k)_q \rangle \quad \forall q \in U.$$

<u>Ex:</u>

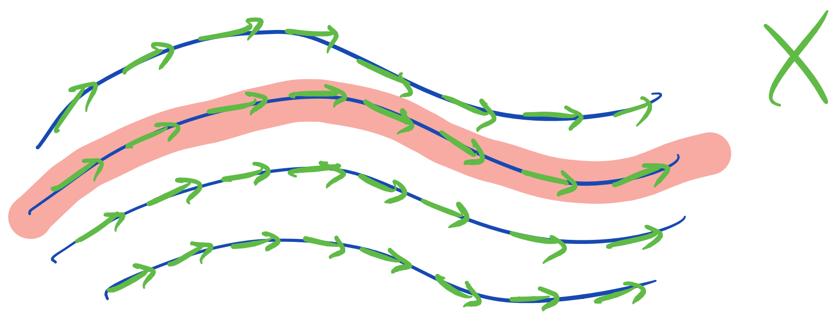


An injective immersion $\varphi : N \hookrightarrow M$ is called an <u>integral submanifold through $q \in M$</u> if $q \in \varphi(N)$ and

$$d_p\varphi(T_pN) = D_{\varphi(p)} \quad \forall p \in N$$

<u>Ex:</u> Every non-vanishing vector field $X \in \mathrm{Vect}(\mathbb{R}^n)$ is a smooth distribution of dimension 1.

An integral submanifold $\varphi: \mathbb{R} \hookrightarrow \mathbb{R}^n$ of X is then nothing but a flow line for X:



<u>Ex</u> (Hopf Fibration): Consider the 3-sphere

$$\mathbb{S}^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\} \subseteq \mathbb{C}^2$$

and the map $p: \mathbb{S}^3 \longrightarrow \mathbb{C} \cup \{\infty\}$ = one pt. compf. $\cong \mathbb{S}^2$

$(z_1, z_2) \longmapsto z_1/z_2$

In polar coordinates:

$$p(r_1 \cdot e^{i\theta_1}, r_2 \cdot e^{i\theta_2}) = \frac{r_1}{r_2} \cdot e^{i(\theta_1 - \theta_2)} \quad ; \quad r_1^2 + r_2^2 = 1$$

The preimage of a circle $\varrho \cdot \mathbb{S}^1 = \{z \in \mathbb{C} \mid |z| = \varrho\}$ is a torus:

$$p^{-1}(\varrho \cdot \mathbb{S}^1) = \left\{\left(\frac{\varrho}{\sqrt{1+\varrho^2}}\, e^{i\theta_1}, \frac{1}{\sqrt{1+\varrho^2}} \cdot e^{i\theta_2}\right) \middle|\ \theta_1, \theta_2 \in \mathbb{R}\right\}$$

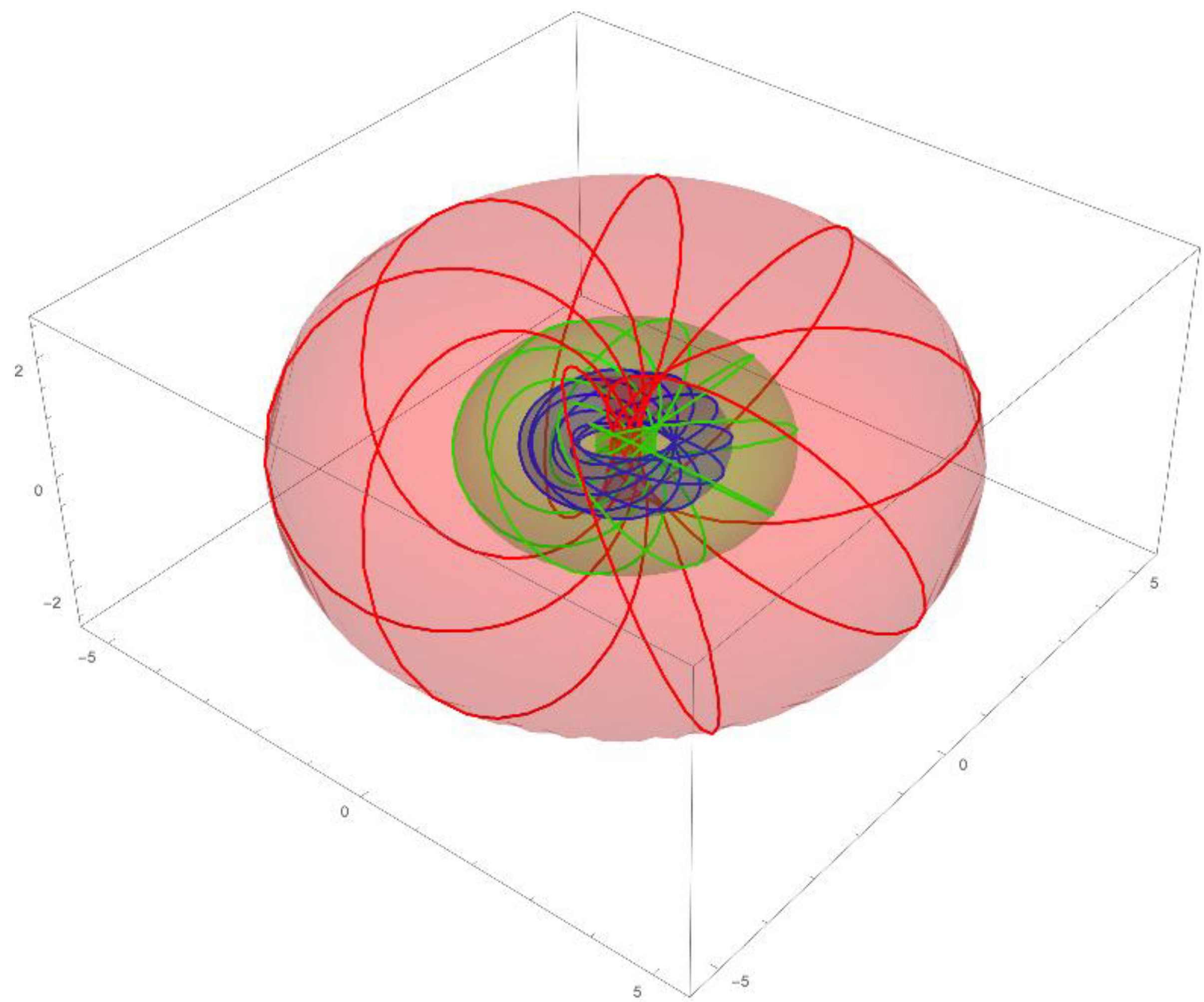
$$\cong \mathbb{S}^1 \times \mathbb{S}^1$$

The preimage of a point $z=\rho\cdot e^{it}$ is a circle:

$$p^{-1}(\rho\cdot e^{it}) = \left\{\left(\frac{\rho}{\sqrt{1+\rho^2}}\, e^{i(\theta+t)}, \frac{1}{\sqrt{1+\rho^2}}\, e^{i\theta}\right) \middle| \theta\in\mathbb{R}\right\}$$
$$\cong S^1$$

Then $\mathcal{D} = \left\{\ker(dp_{(z_1,z_2)})\right\}_{(z_1,z_2)\in S^3}$ is a smooth distribution of dimension 1.

Its integral submanifolds are just its fibers $p^{-1}(z)\cong S^1$. (via stereographic projection:)



$\mathcal{D}$ is called involutive if it is "closed under taking the Lie bracket": $[X_i, X_j] \in \mathcal{D}$

Ex: Any 1-dim. distribution is involutive:

$$[X, X] = 0.$$

$\mathcal{D}$ is called completely integrable if there is an integral submanifold through every point of M. (Some authors call this just "integrable")

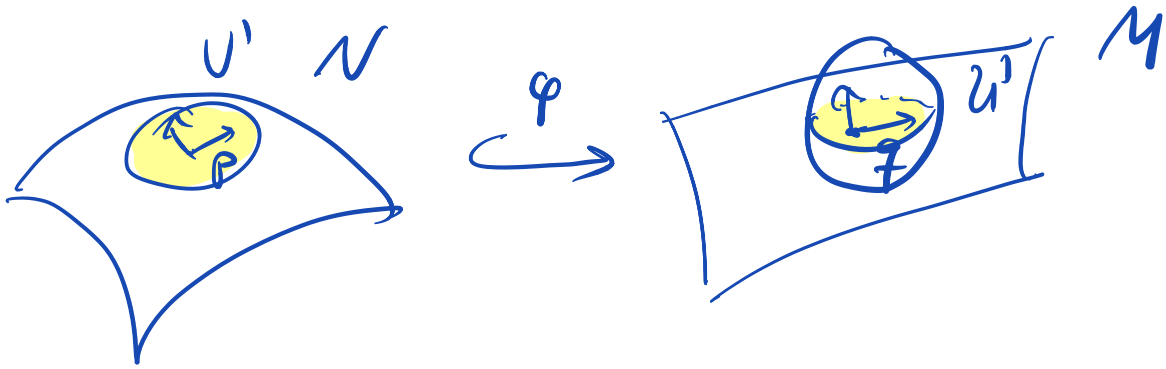
Theorem (Frobenius): A distribution $\mathcal{D}$ is completely integrable if and only if it is involutive.

Ex. 4 shows: completely integrable $\Rightarrow$ involutive.

Exercise 4.(Easy Direction of Frobenius' Theorem):

Let M be a smooth manifold and let $\mathcal{D}$ be a distribution on M. Show that $\mathcal{D}$ is involutive if it is completely integrable.

Solution: Let $U \subseteq M$ open, $\{X_1, \ldots, X_k\}$ local basis for $\mathcal{D}$. Let $q \in U$, and let $\varphi: N \hookrightarrow M$ be an integral submanifold through q; i.e. $\exists p \in N$: $\varphi(p) = q$



There are open nbhds $p \in V' \subset N$, $q \in U' \subset U$

s.t.

$$\varphi|_{V'} : V' \hookrightarrow U' \text{ is a smooth embedding.}$$

Pull-back vector fields:

$$(Y_i)_{p'} := (d\varphi_{p'})^{-1}\big((X_i)_{\varphi(p')}\big) \quad \forall p' \in V'$$

Thus: $Y_i \overset{\varphi}{\sim} X_i$ (φ-related)

Ex 1 $\Longrightarrow$

$$[Y_i, Y_j] \overset{\varphi}{\sim} [X_i, X_j]$$

Hence: $[X_i, X_j]_{\varphi(p)} = d\varphi_p(\underbrace{[Y_i, Y_j]_p}_{\in T_pN}) \in \mathcal{D}_{\varphi(p)}$

□

Exercise 6.(Functions with values in immersed submanifolds):

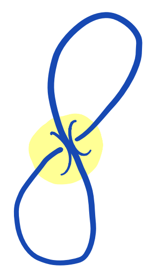
Let M', M, N be smooth manifolds and let $\iota\colon N \hookrightarrow M$ be an injective immersion, i.e. ι is an injective smooth map whose differential is injective. Further, let $f\colon M' \to M$ be a smooth map with $f(M) \subseteq \iota(N)$.

Show that $\iota^{-1} \circ f\colon M' \to N$ is smooth if it is continuous.

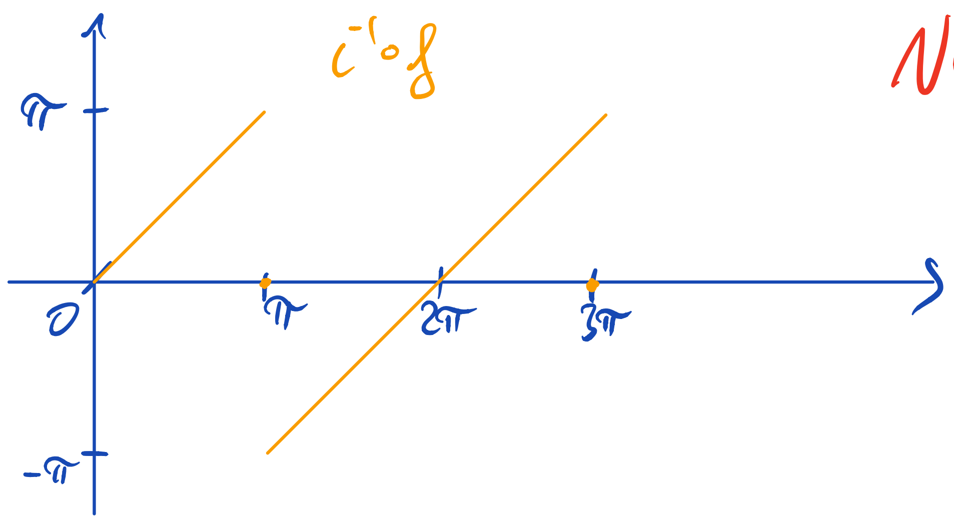
Does not hold without $\iota^{-1} \circ f$ being continuous:

Figure eight curve:

$$\iota : (-\pi, \pi) \hookrightarrow \mathbb{R}^2$$
$$t \longmapsto (\sin(2t), \sin(t))$$

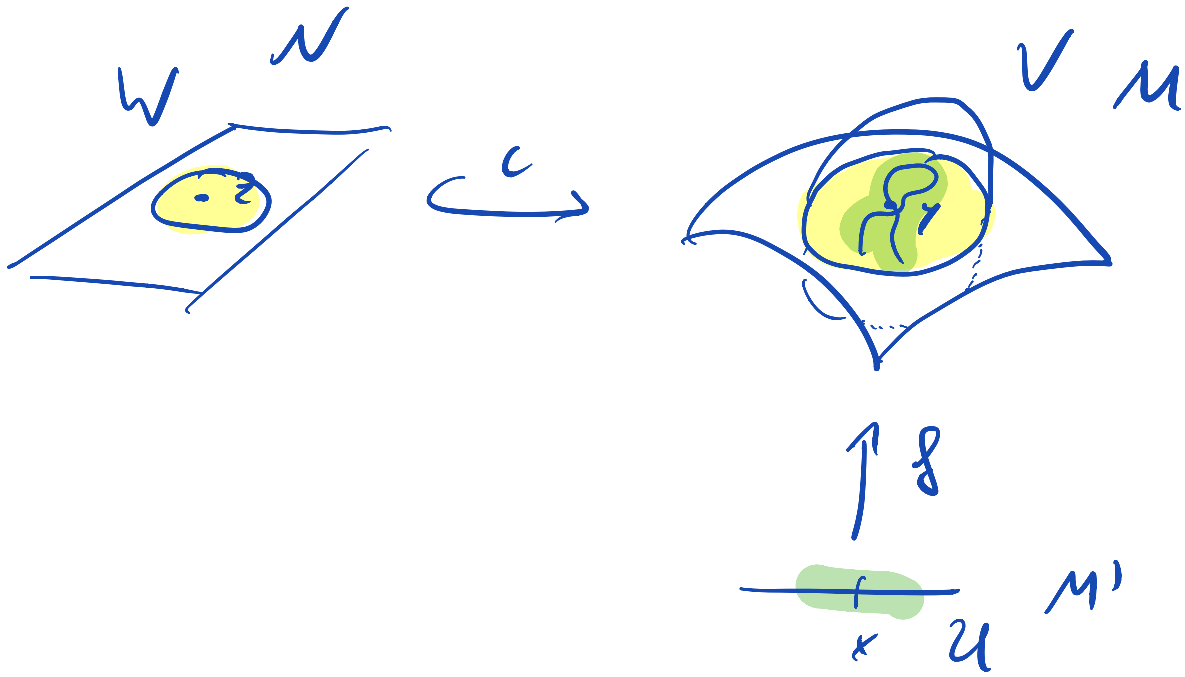


$$f : \mathbb{R} \longrightarrow \mathbb{R}^2$$
$$t \longmapsto (\sin(2t), \sin(t))$$



NOT even continuous!

Solution: Let $x \in M'$, $y = f(x) \in M$, $z = \iota^{-1}(y) \in N$



$\exists\, W \subseteq N$, $V \subseteq M$ open nbhds of z, y resp. & charts $\xi : W \to \mathbb{R}^k$, $\psi : V \to \mathbb{R}^m$ s.t. $\iota(W) \subseteq V$ and

$$j(x_1,\dots,x_k) := (\psi \circ \iota \circ \xi^{-1})(x_1,\dots,x_k)$$
$$= (x_1,\dots,x_k,0,\dots,0)$$

The open (!) set $(\iota^{-1} \circ f)(W) \subseteq M'$ (continuous) contains a chart $\varphi : U \to \mathbb{R}^m$ of x

$$\begin{tikzcd}
W \arrow[r, hook, "\iota|_W"] \arrow[d, "\xi"] & V \arrow[d, "\psi"] & U \arrow[l, "f|_U"'] \arrow[d, "\varphi"] \\
\mathbb{R}^k \arrow[r, bend left, "j"] & \mathbb{R}^k \times \{0\} \arrow[l, bend left, "\pi"] & \mathbb{R}^m, \arrow[l]
\end{tikzcd}$$

$\Rightarrow \iota^{-1} \circ f$ is (locally) smooth. $\square$

Exercise 5.(Distributions and Lie Subalgebras):

b) Show that the Lie algebra $\mathfrak{h}$ of a Lie subgroup H of a Lie group G determines a left-invariant involutive distribution.

Solution: Let $\iota : H \hookrightarrow G$ be an injective imm. hom.

Take a basis $\{v_1, \dots, v_n\} \in T_e H$.

Define $$(X_i)_g := d_e L_g(d_e \iota(v_i)) \quad \forall g \in G.$$

and set $$\mathcal{D} = \{\langle (X_1)_g, \dots, (X_n)_g \rangle \subset T_g G\}$$

By defn $\mathcal{D}$ is left-inv.

Claim: $L_g \circ \iota : H \hookrightarrow G$ is an integral submfld of $\mathcal{D}$ through $g \in G$.

Pf: $$(X_i)_g = d_e L_g(d_e \iota(v_i)) = d_e(L_g \circ \iota)(v_i) \quad \forall i = 1, \dots, n \ \forall g \in G.$$

$\Rightarrow$ $\mathcal{D}$ is completely integrable [For a direct proof see solution]

$\overset{\text{Ex 4}}{\Rightarrow}$ $\mathcal{D}$ is involutive $\square$