25 November 2020

Theorem (van Neumann for GL, Cartan) G real Lie gp, $H<G$ closed subgroup $\Rightarrow$ H is a real Lie group.

Rk False for complex Lie gps.

Lemma 1 Let G be a Lie gp, $\mathrm{Lie}(G)=\mathfrak{g}$, $H<G$ abstract subgp, $\mathfrak{h}\subset\mathfrak{g}$ subspace. Let $\exp: U_0 \to V_e$ be a diffeo and assume

$$(*)\quad \exp(U_0\cap\mathfrak{h}) = V_e\cap H.$$

Then:

(1) H is a Lie sbgp of G with the induced topology;

(2) $\mathfrak{h}\subset\mathfrak{g}$ is a subalgebra;

(3) $\mathfrak{h}=\mathrm{Lie}(H)$.

Lemma 2 G Lie gp, $\mathrm{Lie}(G)=\mathfrak{g}$. If $X,Y\in\mathfrak{g}$, then for t small

$$\exp(tX)\exp(tY)=\exp\left(t(X+Y)+O(t^2)\right),$$

where $\frac{1}{t^2}O(t^2)$ is bdd in a nbd of 0.

Pf of theorem

Want to define $\mathfrak{h}\subset\mathfrak{g}$ subspace that satisfies $(*)$. Let

$$\mathfrak{h}:=\{X\in\mathfrak{g} : \exp(tX)\in H\ \forall t\in\mathbb{R}\}$$

To show:

(1) $\mathfrak{h}$ is a subspace (use lemma 2)

(2) $\mathfrak{h}$ satisfies $(*)$.

(1) $\mathfrak{h}$ closed for multipl. by scalars.

Let $X,Y\in\mathfrak{g}$ $\Rightarrow$ lemma 2

$$(\exp(tX)\exp(tY))^n=\left(\exp\left(t(X+Y)+O(t^2)\right)\right)^n=$$
$$=\exp\left(nt(X+Y)+O(nt^2)\right).$$

Replace t by $\frac{t}{n}$ $\Rightarrow$

$$\left(\exp\left(\tfrac{t}{n}X\right)\exp\left(\tfrac{t}{n}Y\right)\right)^n=\exp\left(t(X+Y)+\tfrac{1}{n}O(t^2)\right)$$

$$\underbrace{\lim_{n\to\infty}\left(\exp\left(\tfrac{t}{n}X\right)\exp\left(\tfrac{t}{n}Y\right)\right)^n}_{\in H\text{ (closed)}}=$$
$$=\exp(t(X+Y))\Rightarrow X+Y\in\mathfrak{h}$$

because H is closed.

(2) We show now that

$$\exp(U_0\cap\mathfrak{h})\overset{(*)}{=}V_e\cap H.$$

By contradiction. Obviously

$\exp(U_0\cap\mathfrak{h})\subseteq V_e\cap H$, so

if $(*)$ did not hold $\forall\, U_0\subset\mathfrak{g}$, and $\forall\, V_e\subset G$ such that

$\exp: U_0\to V_e$ is a diffeo, we could find $h\in V_e\cap H$ but $h\notin\exp(U_0\cap\mathfrak{h})$.

So $\exists$ a sequence $(h_k)_{k\geq 1}\subset H$ such that $h_k\to e$, and $W_0\subset\mathfrak{h}$ s.t. $h_k\notin\exp(W_0)$.

Let $\mathfrak{h}'$ be a complementary subspace of $\mathfrak{h}$, $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{h}'$.

In the previous lecture we saw

$$\exp: N_0\times N_0'\longrightarrow A_e$$
$$(X,X')\longmapsto \exp_{\mathfrak{h}}(X)\exp_{\mathfrak{h}'}(X')$$

is a diffeomorphism.

$\Rightarrow$ $(h_k)\subset H$, $h_k\to e$

so $h_k\in A_e$ for large k.

$\Rightarrow$ for large k

$$h_k=\exp(X_k)\exp(X_k')$$

where $X_k \in N_0 \subset \mathfrak{h}$, $X_k' \in N_0' \subset \mathfrak{h}'$.

i) $h_k \notin \exp(W_0)$ and $X_k \in N_0 \subset W_0 \Rightarrow X_k' \neq 0$

ii) $X_k \in N_0 \subset W_0 \Rightarrow \exp(X_k) \in H$

$\Rightarrow$ since $h_k = \exp(X_k)\exp(X_k')$, then $\exp(X_k') = \exp(-X_k)h_k \in H$.

Putting all of this together, we got

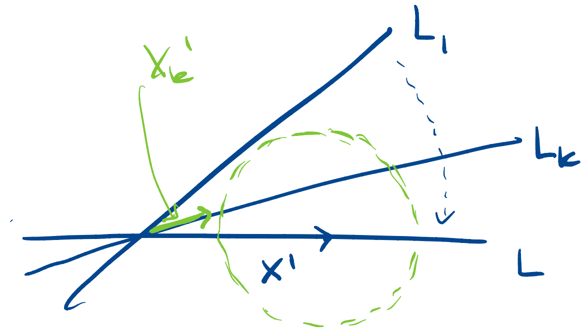
$$e \neq \underbrace{\exp(X_k') \in H}_{(**)} \cap \exp(N_0' \setminus \{0\})$$

Since $h_k \to e$ my X_k' with the properties so far.

Let $L_k := \mathbb{R}X_k' \in \mathbb{P}(\mathfrak{h}')$.

Since $\mathbb{P}(\mathfrak{h}')$ is compact, $L_k \to L \in \mathbb{P}(\mathfrak{h}')$ (perhaps by passing to a subsequence).

Let $X' \in L = \mathbb{R}X'$. Then if $\varepsilon > 0$



for k large enough:

1) $L_k \cap B(X', \varepsilon) \neq \emptyset$

2) $\|X_k'\| < \varepsilon$

3) $\exists\, n_k \in \mathbb{Z}$ s.t.

$\|X' - n_k X_k'\| < \varepsilon$, that is $\lim_{k \to \infty} n_k X_k' = X'$.

But then

$$\exp X' = \lim_{k \to \infty} \exp(n_k X_k') = \lim_{n \to \infty} \underbrace{(\exp(X_k'))^{n_k}}_{\substack{\cap\ (**) \\ H}} \in H,$$

which is impossible since $X' \in L \in \mathbb{P}(\mathfrak{h}')$. ∎

Pause

Adjoint representations

G Lie group, $\mathfrak{g} = \mathrm{Lie}(G)$.

A representation of G over

$\mathbb{k}$ (= $\mathbb{R}$ or $\mathbb{C}$) is a continuous homomorphism $\pi : G \to GL(n, \mathbb{k})$

A repres. of $\mathfrak{g}$ over $\mathbb{k}$ is a homo $\mathfrak{g} \to \mathfrak{gl}(n, \mathbb{k})$.

Fact (seen) Any rep. of G induces by differ. a Lie algebra repr. $d_e\pi : \mathfrak{g} \longrightarrow \mathfrak{gl}(n, \mathbb{k})$

Fact Let $V \subset \mathbb{k}^n$ be a subspace. Then V is $\pi(G)$-invariant $\Leftrightarrow$ it is $d_e\pi(G)$-invariant.

We denote H_V the stab. of V in $GL(n, \mathbb{k})$ and $\mathfrak{h}_V$ its Lie algebra.

We can find coordinates in k^n in such a way that

$$H_v \subset \left\{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\right\} < GL(n,k)$$

$$\mathfrak{h}_v \subset \left\{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\right\} < \mathfrak{gl}(n,k)$$

If $g \in G$, consider $c_g : G \to G$

$c_g(h) = ghg^{-1}$. (in the context of the Haar (m) $c_g(h) = g^{-1}hg$.)

$c_g(e) = e$, $c_g \circ c_h = c_{gh}$ $\Rightarrow$

$\Rightarrow$ $d_e c_g : \mathfrak{g} \to \mathfrak{g}$ is a Lie algebra automorphism.

Defn. Let G be a Lie gp with Lie algebra $\mathfrak{g}$.

(1) The Adjoint representation of G is

$$\mathrm{Ad} : G \longrightarrow GL(\mathfrak{g})$$
$$g \longmapsto d_e c_g$$

(2) The adjoint representation of $\mathfrak{g}$ is

$$\mathrm{ad} : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$$

$$\mathrm{ad} := d_e \mathrm{Ad}.$$

Remark By naturality of the exponential map

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\mathrm{Ad}(g)"] \arrow[d, "\exp_G"'] & \mathfrak{g} \arrow[d, "\exp_G"] \\ G \arrow[r, "c_g"'] & G \end{tikzcd}$$

commutes, that is $\forall X \in \mathfrak{g}$

$$\exp_G \circ \mathrm{Ad}(g)(X) = c_g \circ \exp_G(X)$$

(2) but also

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\mathrm{ad}"] \arrow[d, "\exp_G"'] & \mathfrak{gl}(\mathfrak{g}) \arrow[d, "\exp_{GL(\mathfrak{g})}"] \\ G \arrow[r, "\mathrm{Ad}"'] & GL(\mathfrak{g}) \end{tikzcd}$$

commutes, that is

$$\exp_{GL(\mathfrak{g})} \circ \mathrm{ad} = \mathrm{Ad} \circ \exp_G.$$

Proposition If G is a closed subgroup of $GL(n,\mathbb{R})$, then

$$\mathrm{Ad}(g)(X) = gXg^{-1}$$

for all $g \in G$, $X \in \mathfrak{g}$.

Proof Since $G < GL(n,\mathbb{R})$ $\Rightarrow$

$\Rightarrow d_e c_g = c_g$ $\square$

Proposition If G is a Lie gp with Lie algebra $\mathfrak{g}$, then for all $X, Y \in \mathfrak{g}$,

$$\mathrm{ad}(X)(Y) = [X,Y].$$

Definition Let $\mathfrak{g}$ be a Lie algebra. The adjoint representation of $\mathfrak{g}$ is

$$\mathrm{ad} : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$$
$$X \longmapsto \mathrm{ad}(X),$$

where $\mathrm{ad}(X)(Y) := [X,Y]$.

Pf of Proposition

$\mathfrak{g} = \mathrm{Lie}(G)$, $\mathrm{ad} = d_e \mathrm{Ad}$.

$$\mathrm{ad}(X)(Y) = [X,Y]$$

Sloppy proof. $X, Y \in \mathfrak{g}$

$$ad(X)(Y) = (d_e Ad)(X)(Y) =$$

$$= \frac{d}{dt}\Big|_{t=0} Ad(\exp(tX))(Y)$$

$$= \frac{d}{dt}\Big|_{t=0} (d_e c_{\exp(tX)})(Y)$$

$$= \frac{d}{dt}\Big|_{t=0} \frac{d}{ds}\Big|_{s=0} c_{\exp(tX)}(\exp sY)$$

$$= \frac{d}{dt}\Big|_{t=0} \frac{d}{ds}\Big|_{s=0} \exp(tX)\exp(sY)\exp(-tX)$$

If G is linear $d_e c_g = c_g$

$$\Rightarrow ad(X)(Y) =$$

$$= \frac{d}{dt}\Big|_{t=0} \exp(tX)\, Y \exp(-tX) =$$

$$= XY - YX.$$

We need to consider the left invariant v.f. associated to an element in $\mathfrak{g}$.

Convention If $Z \in \mathfrak{g}$, $\tilde{Z} \in Vect(G)^G$ with $\tilde{Z}_e = Z$.

Naturality of exp

$$Ad(\exp_G(tX)) = \exp_{GL(\mathfrak{g})}(ad(tX))$$

$$= e^{ad(tX)}$$

matrix exponential

If $Y \in \mathfrak{g}$

$$Ad(\exp_G(tX))(Y) = e^{ad(tX)}(Y)$$

$$= 1 + t\, ad(X)(Y) + \frac{t^2}{2} R(t,X)(Y)$$

$$\widetilde{ad(X)(Y)} \overset{(*)}{=} \frac{d}{dt}\Big|_{t=0} \widetilde{Ad(\exp_G(tX))(Y)}$$

To compute the derivative:

$Z \in \mathfrak{g}$, $f \in C^\infty(G)$

$$\boxed{\begin{aligned} \tilde{Z}_g(f) &= (d_g f)(\tilde{Z}_g) = \\ &= (d_g f)(d_e L_g)(Z) = \\ &= d_e(f \circ L_g)(Z). \end{aligned}} \quad (*)$$

(1) $s \mapsto \exp(sZ)$ goes through e at $s=0$ with tg vector Z.

(*) at $g\exp(tX)$

$$\tilde{Z}_{g\exp(tX)}(f) = d_e(f \circ L_{g\exp(tX)})(Z)$$

$$= \frac{d}{ds}\Big|_{s=0} (f \circ L_{g\exp(tX)})(\exp(sZ))$$

$$= \frac{d}{ds}\Big|_{s=0} f(g\exp(tX)\exp(sZ))$$

(2) $Z = Ad(h)(Y) = d_e c_h(Y)$

tg to $s \mapsto h\exp(sY)h^{-1}$

$\Rightarrow$ (*) to $\widetilde{Ad(h)(Y)}$ $\Rightarrow$

$$\Rightarrow \widetilde{Ad(h)(Y)}_g(f) =$$

$$= d_e(f \circ L_g)(Ad(h)(Y)) =$$

$$= \frac{d}{ds}\Big|_{s=0} (f \circ L_g)(h\exp(sY)h^{-1}) =$$

$$= \frac{d}{ds}\Big|_{s=0} f(gh\exp(sY)h^{-1}).$$

From (*)

$$\widetilde{ad(X)(Y)} = \frac{d}{dt}\Big|_{t=0} \widetilde{Ad(\exp_G(tX))(Y)}$$

(2) $h = \exp(tX)$

$$= \frac{d}{dt}\Big|_{t=0} \frac{d}{ds}\Big|_{s=0} f(g\exp(tX)\exp(sY)\exp(-tX))$$

$=$

Ex. If $F(u,v)$ is diff. $\Rightarrow$

$$\frac{d}{dt}\Big|_{t=0} F(t,t) = \frac{d}{dt}\Big|_{t=0} F(0,t) + \frac{d}{dt}\Big|_{t=0} F(t,0).$$

$$= \frac{d}{ds}\Big|_{s=0} f(g\exp(tX)\exp(sY)) + \frac{d}{ds}\Big|_{s=0} f(g\exp(sY)\exp(-tX))$$

$$\overset{(1)}{=} \frac{d}{dt}\Big|_{t=0} \tilde{Y}_{g\exp(tX)}(f) - \frac{d}{ds}\Big|_{s=0} \tilde{X}_{g\exp(sY)}(f) =$$

$$= (\tilde{X}\circ\tilde{Y})_g(f) - (\tilde{Y}\circ\tilde{X})_g(f) = [\tilde{X},\tilde{Y}]_g(f)$$

$$= [X,Y]_g(f) \Rightarrow$$

$\Rightarrow$ at $g=e$

$$\boxed{ad(X)(Y) = [X,Y]}.$$