26 November 2020

Last time: $\mathrm{ad}: \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$ and $\mathrm{ad}(\mathfrak{g}) \subset \mathfrak{gl}(\mathfrak{g})$. But in fact $\mathrm{ad}(\mathfrak{g}) \subset \mathrm{Der}(\mathfrak{g})$.

$\delta \in \mathrm{Der}(\mathfrak{g}) \Rightarrow \delta[X,Y] = [\delta X, Y] + [X, \delta Y]$ $\forall X,Y \in \mathfrak{g}$ and Jacobi identity

$\Rightarrow \mathrm{ad}(X)[Y,Z] = [\mathrm{ad}(X)(Y), Z] + [Y, \mathrm{ad}(X)(Z)]$

Defn. (1) $\mathfrak{g}$ Lie algebra. An <u>ideal</u> $\mathfrak{h} \subset \mathfrak{g}$ is a subspace s.t. $\forall X \in \mathfrak{g}$ and $Y \in \mathfrak{h}$, $[X,Y] \in \mathfrak{h}$.

(2) An ideal $\mathfrak{h}$ is <u>characteristic</u> if $\delta(\mathfrak{h}) \subset \mathfrak{h}$ $\forall \delta \in \mathrm{Der}(\mathfrak{g})$.

Proposition G conn. Lie gp, $\mathrm{Lie}(G) = \mathfrak{g}$, $H < G$ closed sbgp with $\mathrm{Lie}(H) = \mathfrak{h}$. Then $H \trianglelefteq G \iff \mathfrak{h} \subset \mathfrak{g}$ ideal

Pf H normal $\iff c_g(H) = H$ $\forall g \in G$

$\iff \mathrm{Ad}(g)(\mathfrak{h}) = \mathfrak{h}$ $\forall g \in G$

$\iff \mathrm{ad}(X)(\mathfrak{h}) = \mathfrak{h}$ $\forall X \in \mathfrak{g}$

$\iff \mathfrak{h}$ is an ideal. ∎

Proposition G connected Lie gp. Then $Z(G) = \ker \mathrm{Ad}$ and $Z(\mathfrak{g}) = \ker \mathrm{ad}$

Pf Naturality of $\exp_G \Rightarrow$

$\Rightarrow g \exp(X) g^{-1} = \exp(\mathrm{Ad}(g) X)$.

If $\exp$ is surjective $\Rightarrow$

$g \in Z(G) \iff g \exp(X) g^{-1} = \exp(X)$ $\forall X \in \mathfrak{g}$

$\iff \mathrm{Ad}(g) X = X$ $\forall X \in \mathfrak{g}$.

If exp not surjective the same proof holds since G is generated by elems. were exp is a diffeo. ∎

Lemma If $\mathfrak{k} \subset \mathfrak{g}$ is an ideal and $\mathfrak{h} \subset \mathfrak{k}$ is a char. ideal $\Rightarrow \mathfrak{h} \subset \mathfrak{g}$ ideal.

Pf If $X \in \mathfrak{g} \Rightarrow \delta_X(Y) := [X,Y]$ is a derivation of $\mathfrak{g}$ $(\delta_X(Y) = \mathrm{ad}(Y))$. Since $\mathfrak{h} \subset \mathfrak{k}$ is a char. ideal $\delta_X(Y) = [X,Y] \in$ $\in \mathfrak{h}$ $\forall Y \in \mathfrak{h}, X \in \mathfrak{g} \Rightarrow$

$\Rightarrow \mathfrak{h}$ is an ideal in $\mathfrak{g}$. ∎

Example

(1) $[\mathfrak{g},\mathfrak{g}]$ is a char. ideal in $\mathfrak{g}$ since if $\delta \in \mathrm{Der}(\mathfrak{g}) \Rightarrow$

$\Rightarrow \delta[X,Y] = [\delta X, Y] + [X, \delta Y] \in$

$\in [\mathfrak{g},\mathfrak{g}] = \mathrm{span}\{[X,Y] : X,Y \in \mathfrak{g}\}$.

(2) $\mathfrak{g}^{(1)} := [\mathfrak{g},\mathfrak{g}]$

$\mathfrak{g}^{(i+1)} := [\mathfrak{g}^{(i)}, \mathfrak{g}^{(i)}]$

$\Rightarrow \mathfrak{g}^{(i+1)}$ is a char. ideal of $\mathfrak{g}^{(i)} \Rightarrow \mathfrak{g}^{(i+1)}$ is an ideal of $\mathfrak{g}$. ∎

Defn. (Solvable Lie algebra) The chain of ideals

$$\mathfrak{g} \supseteq \mathfrak{g}^{(1)} \supseteq \dots \supseteq \mathfrak{g}^{(n)} \supseteq \dots$$

is called the derived series of $\mathfrak{g}$.

A Lie algebra is solvable if $\mathfrak{g}^{(k)} = \{0\}$ for some k.

Example Abelian $\Rightarrow$

$\Rightarrow$ Solvable.

Proposition Let $\mathfrak{g}$ be a Lie alg. $\mathfrak{g}$ is solvable $\iff$ $\exists$ a chain of subalgebras

(*) $\mathfrak{g} \supset \mathfrak{g}_1 \supset \dots \supset \mathfrak{g}_n = \{0\}$ s.t.
i) $\mathfrak{g}_{i+1}$ is an ideal in $\mathfrak{g}_i$
ii) $\mathfrak{g}_i / \mathfrak{g}_{i+1}$ Abelian

Pf ($\implies$) Take the derived series, $\mathfrak{g}_i := \mathfrak{g}^{(i)}$. (i) is verified $\mathfrak{g}_i/\mathfrak{g}_{i+1} = \mathfrak{g}^{(i)} / [\mathfrak{g}^{(i)}, \mathfrak{g}^{(i)}]$

$\implies$ (ii) is verified as well.

($\impliedby$) Assume $\mathfrak{g}_i$ exists as in (*) exist and show that the derived series terminates, by

showing that $\mathfrak{g}^{(i)} \subset \mathfrak{g}_i$.

$n=1$ $\mathfrak{g} \supset \mathfrak{g}_1 = \{0\} \implies \mathfrak{g}$ Abelian

$n=2$ $\mathfrak{g} \supset \mathfrak{g}_1 \supset \mathfrak{g}_2 = \{0\}$, where $\mathfrak{g}/\mathfrak{g}_1$ is Abelian $\implies$

$\implies [\mathfrak{g}/\mathfrak{g}_1, \mathfrak{g}/\mathfrak{g}_1] = 0$ in $\mathfrak{g}/\mathfrak{g}_1$

(*) $\implies \underbrace{[\mathfrak{g},\mathfrak{g}]}_{= \mathfrak{g}^{(1)}} \subset \mathfrak{g}_1$.

$n>2$ Suppose $\mathfrak{g}_{n-1} \supset \mathfrak{g}^{(n-1)}$

$\mathfrak{g}^{(n)} = [\mathfrak{g}^{(n-1)}, \mathfrak{g}^{(n-1)}] \subset$

$\subset [\mathfrak{g}_{n-1}, \mathfrak{g}_{n-1}] \subset \mathfrak{g}_n$ since $\mathfrak{g}_n / \mathfrak{g}_{n-1}$ is Abelian (see (*)) $\blacksquare$

Corollary $\mathfrak{g}$ Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ ideal. Then $\mathfrak{g}$ solvable $\iff$ $\mathfrak{h}$, $\mathfrak{g}/\mathfrak{h}$ solvable

Remark The class $\mathcal{S}$ of solvable Lie algebras is the smallest class s.t.
(1) Abelian algebras $\subset \mathcal{S}$
(2) If $\{0\} \hookrightarrow \mathfrak{h} \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{h} \to \{0\}$ and $\mathfrak{h}, \mathfrak{g}/\mathfrak{h} \in \mathcal{S} \implies \mathfrak{g} \subset \mathcal{S}$.

Pf of corollary
($\implies$) Obvious once one knows that $\mathrm{Lie}(G/H) = \mathrm{Lie}(G)/\mathrm{Lie}(H)$

($\impliedby$) Let $\mathfrak{g}/\mathfrak{h} \supset \mathfrak{k}_1 \supset \mathfrak{k}_2 \supset \dots \supset \mathfrak{k}_k = \{0\}$ s.t. $\mathfrak{k}_i/\mathfrak{k}_{i+1}$ is Abelian,

$\mathfrak{h} \supset \mathfrak{h}_1 \supset \dots \supset \mathfrak{h}_n = \{0\}$ s.t. $\mathfrak{h}_j/\mathfrak{h}_{j+1}$ is Abelian, let $p: \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ be the quotient map and

$\mathfrak{g}_i := p^{-1}(\mathfrak{k}_i)$ for $1 \le i \le k$

and $\mathfrak{g}_j := \mathfrak{h}_{j-k}$ for $k \le j \le k+n$ $\implies$

$\mathfrak{g} \supset p^{-1}(\mathfrak{k}_1) \supset \dots \supset p^{-1}(\mathfrak{k}_k) = \mathfrak{h} \supset$
$\supset \mathfrak{h}_1 \supset \dots \supset \mathfrak{h}_n = \{0\}$. $\blacksquare$

Example

$$\mathfrak{n} = \left\{ \begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix} \in \mathfrak{gl}(n,\mathbb{R}) \right\}$$

$$\mathfrak{n}^{(1)} = \left\{ \begin{pmatrix} 0 & 0 & & * \\ & \ddots & \ddots & \\ 0 & & & 0 \end{pmatrix} \in \mathfrak{gl}(n,\mathbb{R}) \right\}$$

$$\mathfrak{n}^{(2)} = \left\{ \begin{pmatrix} 0 & 0 & 0 & & * \\ & \ddots & \ddots & \ddots & \\ 0 & & & & 0 \end{pmatrix} \in \mathfrak{gl}(n,\mathbb{R}) \right\}$$

$\Rightarrow$ $\mathfrak{n}$ is solvable.

Defn. (Solvable Lie gp)

G connected Lie gp. G is solvable if $\mathrm{Lie}(G)$ is solvable.

Ex.

$$N = \left\{ \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \in GL(n,\mathbb{R}) \right\}$$

is solvable

Proposition G connected Lie group. G is solvable $\iff$ $\exists$ chain of closed sbgps

$$G \geq G_1 \geq \ldots \geq G_k = \{e\}$$

s.t.

(i) $G_{i+1} \trianglelefteq G_i$

(ii) G_i / G_{i+1} Abelian

Pf ($\Leftarrow$) Obvious by taking the Lie algebra of these Lie groups.

($\Rightarrow$) G solvable $\iff$ $\mathfrak{g}$ solvable

$$\mathfrak{g} \supset \mathfrak{g}^{(1)} \supset \mathfrak{g}^{(2)} \supset \ldots \supset \mathfrak{g}^{(k)} = \{0\}.$$

Idea: replace at each step each ideal with one with the same properties but s.t. the corresp. gp. is closed.

Start with $\mathfrak{g}^{(1)}$. Let G_1 be the conn. Lie gp. s.t. $\mathrm{Lie}(G_1) = \mathfrak{g}^{(1)}$. Let $\overline{G}_1$ be the top. closure of G_1 and $\overline{\mathfrak{g}}_1 = \mathrm{Lie}(\overline{G}_1)$. $\Rightarrow$

$$\Rightarrow [\mathfrak{g},\mathfrak{g}] = \mathfrak{g}^{(1)} \subset \overline{\mathfrak{g}}_1 \subset \mathfrak{g}$$

$\Rightarrow$ $\mathfrak{g}/\overline{\mathfrak{g}}_1$ is Abelian

since there is an onto

map $\mathfrak{g}/\mathfrak{g}^{(1)} \twoheadrightarrow \mathfrak{g}/\overline{\mathfrak{g}}_1$.

Also $\mathfrak{g}^{(1)}$ ideal $\Rightarrow$

$\Rightarrow$ $\overline{\mathfrak{g}}_1$ ideal. $\Rightarrow$

$$\mathfrak{g} \supset \overline{\mathfrak{g}}_1 \supset \mathfrak{g}^{(2)} \supset \ldots \supset \mathfrak{g}^{(k)} = \{0\}.$$

Now ? , that is we need to check:

(i) $\mathfrak{g}^{(2)}$ ideal in $\overline{\mathfrak{g}}_1$

(obvious since $\mathfrak{g}^{(2)}$ is an ideal in $\mathfrak{g} \supset \overline{\mathfrak{g}}_1$)

(ii) $\overline{\mathfrak{g}}_1/\mathfrak{g}^{(2)}$ is Abelian.

$\overline{\mathfrak{g}}_1/\mathfrak{g}^{(2)}$ Abelian $\iff$ the $\overline{\mathfrak{g}}_1$-action on $\overline{\mathfrak{g}}_1/\mathfrak{g}^{(2)}$ via ad is trivial.

In fact

$\bar{\mathfrak g}_1/\mathfrak g^{(2)}$ Abelian $\iff$

$\iff [\bar{\mathfrak g}_1, \bar{\mathfrak g}_1] \subset \mathfrak g^{(2)}$

$\iff [X,Y] \in \mathfrak g^{(2)} \quad \forall X,Y \in \bar{\mathfrak g}_1$

$\iff [X, Y + \mathfrak g^{(2)}] \in \mathfrak g^{(2)} \quad \forall X,Y \in \bar{\mathfrak g}_1$

$\iff \mathrm{ad}(\bar{\mathfrak g}_1)(\bar{\mathfrak g}_1/\mathfrak g^{(2)}) \subset \mathfrak g^{(2)}$

True since $\mathfrak g^{(2)}$ is an ideal in $\bar{\mathfrak g}_1$. But $\mathfrak g^{(2)}$ is also an ideal in $\mathfrak g^{(1)} \Rightarrow$

$\mathfrak g^{(1)}/\mathfrak g^{(2)}$ Abelian $\iff$

$\iff \mathrm{ad}(\mathfrak g^{(1)})(\mathfrak g^{(1)}/\mathfrak g^{(2)}) \subset \mathfrak g^{(2)}$.

$\mathfrak g^{(1)}/\mathfrak g^{(2)}$ Abelian $\iff$

$\mathfrak g^{(1)}$-action on $\mathfrak g^{(1)}/\mathfrak g^{(2)}$ via ad trivial $\iff$

G_1-action on $\mathfrak g^{(1)}/\mathfrak g^{(2)}$ via Ad trivial $\iff$

$\bar G_1$-action on $\mathfrak g^{(1)}/\mathfrak g^{(2)}$ via Ad trivial $\iff$

$\bar{\mathfrak g}_1$-action on $\mathfrak g^{(1)}/\mathfrak g^{(2)}$ via ad trivial $\iff$

$\mathfrak g^{(1)}$-action on $\bar{\mathfrak g}_1/\mathfrak g^{(2)}$ via ad trivial $\iff$

G_1 $\quad \bar{\mathfrak g}_1/\mathfrak g^{(2)}$ Ad

$\bar G_1$ $\quad \bar{\mathfrak g}_1/\mathfrak g^{(2)}$ Ad

$\bar{\mathfrak g}_1$ $\quad \bar{\mathfrak g}_1/\mathfrak g^{(2)}$ ad

$\bar{\mathfrak g}_1/\bar{\mathfrak g}_2$ is Abelian. ∎

Lie Theorem

(1) G connected solvable Lie gp and $\pi: G \to GL(n,\mathbb C)$ a complex representation. Then $\exists$ basis of $\mathbb C^n$ w.r.t. to which

$$\pi(G) \le \left\{ \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} < GL(n,\mathbb C) \right\}$$

(2) $\mathfrak g$ solvable Lie algebra, $\rho: \mathfrak g \to \mathfrak{gl}(n,\mathbb C)$ a cplx representation. Then $\exists$ basis of $\mathbb C^n$ w.r.t. which $\rho(\mathfrak g)$ consists of upper triangular matrices.

Defn V $\mathbb C$-vector space. A *full flag* is a chain of subspaces

$\{0\} \subseteq V_1 \subseteq \dots \subseteq V_n$ s.t. $\dim V_j = j$. (i.e. $\dim_{\mathbb C} V = n$)

The flag is G-invariant ($\mathfrak g$-inv.) if $G(V_i) \subset V_i$ for all i

(I should have written $\pi: G \to GL(V)$, $\pi(G)V_i \subset V_i$, $\rho: \mathfrak g \to \mathfrak{gl}(V)$ ___)

Corollary V v.s. over $\mathbb C$ with $\dim_{\mathbb C} V = n$, $G < GL(V)$ Lie gp with $\mathrm{Lie}(G) = \mathfrak g$.

TFAE:

(1) G is solvable ($\mathfrak g$ is solv.)

(2) $\exists$ G-inv. full flag ($\mathfrak g$-inv.)

(3) $\exists$ basis of V s.t. G (or $\mathfrak g$) $\subset \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} < GL(V) \right\}$.

Lemma Let $\pi: G \to GL(V)$ repr. of G and $d_e\pi: \mathfrak{g} \to \mathfrak{gl}(V)$. A vector $v \in V$ is a joint eigenvector for $\{\pi(g) : g \in G\}$ iff it is a joint eigenvector for $\{d_e\pi(X) : X \in \mathfrak{g}\}$.

- v joint e. vector of $\{\pi(g) : g \in G\}$ if
  $\pi(g) v = \chi(g) v$, where
  $\chi : G \to \mathbb{C}^*$ smooth homo
- v joint e.vector of $\{d_e\pi(X) : X \in \mathfrak{g}\}$ if
  $d_e\pi(X) v = \lambda(X) v$, where
  $\lambda \in (\mathfrak{g})^*$.

Moreover
$$\chi(\exp(X)) = e^{\lambda(X)} \quad \forall X \in \mathfrak{g}.$$