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Goal:
- Prove Lie's thm
- see some conseq.
- (nilpotent)

Lie's thm (1) G connected solvable Lie gp, $\pi: G \to GL(n,\mathbb{C})$ complex repr. $\Rightarrow \exists$ basis of $\mathbb{C}^n$ s.t.

$$\pi(G) \leq \left\{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\right\} \leq GL(n,\mathbb{C})$$

(2) $\mathfrak{g}$ solvable Lie algebra, $\rho: \mathfrak{g} \to \mathfrak{gl}(n,\mathbb{C})$ complex repr $\Rightarrow$
$\Rightarrow \exists$ basis of $\mathbb{C}^n$ s.t.

$$\rho(\mathfrak{g}) \subseteq \left\{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\right\} \subset \mathfrak{gl}(n,\mathbb{C}).$$

Preliminaries

(L1) • let $v \in \mathbb{C}^n \setminus \{0\}$ be s.t.
$\pi(g)v = \chi(g)v \quad \forall\, g \in G.$
Then v is a common eigenvector of π.
π invertible homo $\Rightarrow$
$\Rightarrow \chi$ invertible homo

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$\chi \in \mathrm{Hom}(G, \mathbb{C}^*)$.

• let $v \in \mathbb{C}^n \setminus \{0\}$ be s.t.
$\rho(X)v = \lambda(X)\, v \quad \forall\, X \in \mathfrak{g} \Rightarrow$
$\Rightarrow v$ is a common eigenvector of ρ.
ρ Lie alg. repr $\Rightarrow \lambda: \mathfrak{g} \to \mathbb{C}$ is a linear map.

Q: If $\rho = d_e\pi$ is there a rel. between c.e. of π & c.e. of ρ; and what about χ & λ?

Lemma v is common eigenv. of π iff v is a common e.v. of $d_e\pi$. Moreover
$$\chi(\exp X) = e^{\lambda(X)} \quad \forall\, X \in \mathfrak{g}.$$

Idea of proof

$\Rightarrow$ obvious by diff.

$\Leftarrow$ let v be a common e.v. of $d_e\pi$.

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let $G_v := \{g \in G : \pi(g)\mathbb{C}v = \mathbb{C}v\}$: this is a closed sbgp of G and one can see that $\mathrm{Lie}(G_v) = \mathfrak{g}$. ☒

(L2) $H \trianglelefteq G$ connected normal sbgp $\Rightarrow G \curvearrowright H$ by conjugation $\Rightarrow G \curvearrowright \mathrm{Hom}(H, \mathbb{C}^*)$ via
$$(g \cdot \chi)(h) := \chi(c_g^{-1}(h)) = \\ = \chi(g^{-1}hg)$$

One can give $\mathrm{Hom}(H,\mathbb{C}^*)$ a topology w.r.t. the G-action is continuous.
If the G-orbit of $\chi \in \mathrm{Hom}(H,\mathbb{C}^*)$ is finite $\Rightarrow \chi$ is a fixed pt.

(L3) **Lemma** G solvable $\Rightarrow \exists$ closed connected codim. 1 normal subgroup.

Pf G solvable $\Rightarrow \exists\, G_1 \trianglelefteq G$ closed s.t. G/G_1 Abelian $\Rightarrow G/G_1 \cong \mathbb{R}^n \times \mathbb{T}^k$

let $H_1 \trianglelefteq G/G_1$ a codim. one closed normal subgp.

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If such H_1 did not exist $\Rightarrow$
$\Rightarrow G/G_1$ would be of dim. 1 & hence G_1 is a codim. 1 closed conn. normal subgp. let $p: G \to G/G_1$ and set $H := p^{-1}(H_1)$. Then $H \trianglelefteq G$, closed & connected. By a dimension count on the tg spaces of the underlying mfds $\Rightarrow H$ is codim. 1. ☒

Pf of Lie's thm (for Lie gps).

Enough to show that $\exists$ common eigenvector $v \in \mathbb{C}^n \setminus \{0\}$. $\Rightarrow$

$$\pi(G) \leq \left\{\begin{array}{c} v \\ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \end{array}\right\} \leq GL(n,\mathbb{C})$$

and continue by considering the repr. on $\mathbb{C}^n/\mathbb{C}v$.

Induction on $\dim G$.

$\dim G = 1$ Any cplx. matrix X has an eigenvector $\Rightarrow \mathfrak{g} = \mathbb{R}X$ has a common e.v. $\xRightarrow{(L1)}$ this

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is also a common eigenvector of π.

$\dim G \geq 1$ By (L3) $\exists$ closed conn. normal sbgp $H \trianglelefteq G$ (of codim. 1).

By inductive hyp. $\dim H < \dim G$ $\pi|_H$ has a common eigenvector v,

$\pi(h)v = \chi(h)v \quad \forall h \in H, \ \chi \in \mathrm{Hom}(H, \mathbb{C}^*)$

$V_\chi := \{ v \in \mathbb{C}^n : \pi(h)v = \chi(h)v \ \forall h \in H \}$
$\neq \{0\}$.

- $V_{\chi_1} \neq V_{\chi_2}$ if $\chi_1 \neq \chi_2$.
- $\mathbb{C}^n = \sum_{j<\infty}^{\oplus} V_{\chi_j} \Rightarrow$ $\exists$ only a finite number of $V_\chi \neq \{0\}$.
- $\pi(g) V_\chi = V_{g \cdot \chi}$. In fact,

let $v \in V_\chi$ (that is $\pi(h)v = \chi(h)v$).

Want to see that
$\pi(h)(\pi(g)v) = (g \cdot \chi)(h)(\pi(g)v)$.
In fact

$$\pi(h)\pi(g)v = \pi(g)(\pi(g^{-1})\pi(h)\pi(g)v) =$$
$$= \pi(g)\,\pi(c_g^{-1}(h))v =$$
$$= \pi(g)\,\chi(c_g^{-1}(h))v =$$
$$= \chi(c_g^{-1}(h))\,\pi(g)v =$$
$$= (g \cdot \chi)(h)\,\pi(g)v.$$

$\Rightarrow$ V_χ is G-invariant.

Since H is codim 1 $\Rightarrow$
$\Rightarrow \mathfrak{g} = \mathbb{R}X \oplus \mathfrak{h}$, $\mathfrak{h} = \mathrm{Lie}(H)$.

Consider now
$d_e\pi : \mathfrak{g} \longrightarrow \mathfrak{gl}(V_\chi)$
(since V_χ is $\mathfrak{g}$-invariant)

Since X acting on V_χ has an eigenvalue and

each $v \in V_\chi$ is already an e.v. of $\mathfrak{h}$ $\Rightarrow$ v e.v. of $\mathfrak{g}$. ∎

Corollary (1) Every finite dim. irred. complex represent. of a solvable Lie gp has dim. 1.

(2) Every finite dim. irreducible real repr. of a solv. Lie gp. has dim. at most 2.

Defn. $\mathfrak{g}$ Lie alg. over $\mathbb{R}$.
The complexif. $\mathfrak{g}^{\mathbb{C}}$ of $\mathfrak{g}$ is
$$\mathfrak{g}^{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g} = \mathfrak{g} + i\mathfrak{g},$$
with bracket induced by the one on $\mathfrak{g}$.

Rk $\{X_1, \dots, X_n\}$ basis of $\mathfrak{g}$,
$\Rightarrow \{1 \otimes X_1, \dots, 1 \otimes X_n\}$ basis of $\mathfrak{g}^{\mathbb{C}}$ over $\mathbb{C}$ $\Rightarrow$

$\dim_{\mathbb{C}} \mathfrak{g}^{\mathbb{C}} = \dim_{\mathbb{R}} \mathfrak{g}$ but
$\dim_{\mathbb{R}} \mathfrak{g}^{\mathbb{C}} = \dim_{\mathbb{R}} \mathbb{C} \times \dim_{\mathbb{R}} \mathfrak{g} = 2 \dim_{\mathbb{R}} \mathfrak{g}$.

Corollary $\mathfrak{g}$ solvable $\Leftrightarrow$
$\Leftrightarrow$ $\mathrm{ad}(\mathfrak{g}^{\mathbb{C}})$ is upper triang. w.r.t. to some basis $\{1 \otimes X_1, \dots, 1 \otimes X_n\}$, where $\{X_1, \dots, X_n\}$ basis of $\mathfrak{g}$.

Pf ($\Longrightarrow$) $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} + i\mathfrak{g}$ also solvable
$\Rightarrow \mathrm{ad} : \mathfrak{g}^{\mathbb{C}} \longrightarrow \mathfrak{gl}(\mathfrak{g}^{\mathbb{C}})$ can be realized as upper triang.
(Lie's thm)

($\Longleftarrow$) Upper triang $\Rightarrow$ solvable.
Hence $\mathrm{ad}(\mathfrak{g}^{\mathbb{C}})$ solvable. But
$\mathrm{ad}(\mathfrak{g}^{\mathbb{C}}) = \mathrm{ad}(\mathfrak{g}) + i\,\mathrm{ad}(\mathfrak{g})$
$\Rightarrow \mathrm{ad}(\mathfrak{g})$ solvable (since it is a sub. of $\mathrm{ad}(\mathfrak{g}^{\mathbb{C}})$).

But $0 \longrightarrow Z(\mathfrak{g}) \hookrightarrow \mathfrak{g} \longrightarrow \mathrm{ad}(\mathfrak{g}) \to 0$

$\Rightarrow \mathfrak{g}$ is solvable since $\mathrm{ad}(\mathfrak{g})$ and $Z(\mathfrak{g})$ are. $\blacksquare$

Application $\exists$ Lie gps with no faithful representations.

$$N = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x,y,z \in \mathbb{R} \right\}$$

$$Z(N) = \left\{ \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : z \in \mathbb{R} \right\} =: H \trianglelefteq N$$

$$D = H \cap SL(3,\mathbb{Z}) = \left\{ \begin{pmatrix} 1 & 0 & n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\}$$

$$G := N/D = \left\{ \begin{pmatrix} 1 & x & t \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : \begin{matrix} x,y \in \mathbb{R} \\ t \in S^1 \end{matrix} \right\}$$

We'll show that G has no faithful representations,

because $\pi(H/D) = \mathrm{Id} \in GL(n,\mathbb{C})$ $\forall\, \pi : G \to GL(n,\mathbb{C})$.

Here $H/D = \left\{ \begin{pmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : t \in S^1 \right\}$

Claim 1 $\pi(H/D) \leq \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \right\} =: L$

Claim 2 L cannot have non-trivial compact subgps.

($\Rightarrow \pi(H/D) = \mathrm{Id}$ $\forall \pi$)

Pf of claim 2 If $K \leq L$ is cpt, we'll show that K can be conj. into any nbd of $\mathrm{Id} \in GL(n,\mathbb{C})$, contradicting the no-small-sbgp property of the Lie gp L.

Let $g = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$, $0 < \lambda_1 < \lambda_2 < \dots < \lambda_n$

If $i < j$

$$\left(c_g \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \right)_{ij} = \frac{\lambda_i}{\lambda_j} \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_{ij}$$

$$\left(c_g^n \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \right)_{ij} = \left(\frac{\lambda_i}{\lambda_j} \right)^n \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_{ij}$$

But $\frac{\lambda_i}{\lambda_j} < 1$ and if $\begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \in K$

$\Rightarrow$ it is bdd. $\Rightarrow$

$$\Rightarrow \left(c_g^n \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right)_{ij} \to 0 \text{ unf.}$$

and $c_g^n \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \to \mathrm{Id}$ unf

Pf of Claim 1

$$H/D = \left\{ \begin{pmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : t \in S^1 \right\}$$

In order to see that $\pi(H/D) \leq L$ it is enough to see that

$$d_e\pi(\mathrm{Lie}(H/D)) \subseteq \left\{ \begin{pmatrix} 0 & * & * \\ & \ddots & * \\ 0 & & 0 \end{pmatrix} \right\} \subseteq \mathfrak{gl}(n,\mathbb{C})$$

$$\mathrm{Lie}(H/D) \underset{\substack{\uparrow \\ D \text{ discrete}}}{=} \mathrm{Lie}(H) =: \mathfrak{h}$$

$\mathfrak{h}$ Abelian $\overset{\text{Lie}}{\Longrightarrow}$ $d_e\pi(\mathfrak{h})$ upper triangular.

$$\left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\}$$

$$\mathfrak{n} = \mathrm{Lie}(N) = \mathrm{Lie}(G) = \left\{ \begin{pmatrix} 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix} \right\}$$

$\left(\text{since } N = \left\{ \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \right\} \right)$

$\Rightarrow$ If $\rho := d_e\pi$, by Lie

$\rho(\mathfrak{r})$ is upper triangular

$$\rho(\mathfrak{r}) \subseteq \left\{ \begin{pmatrix} * & * & * \\ & * & * \\ 0 & & * \end{pmatrix} \right\} \subseteq \mathfrak{gl}(n,\mathbb{C})$$

$$[\rho(\mathfrak{r}), \rho(\mathfrak{r})] \subseteq \left\{ \begin{pmatrix} 0 & * & * \\ & & * \\ 0 & & 0 \end{pmatrix} \right\} \subseteq \mathfrak{gl}(n,\mathbb{C})$$

$$\| \quad \rho([\mathfrak{r},\mathfrak{r}]) . \text{ But}$$

$$\boxed{[\mathfrak{r},\mathfrak{r}] = \mathfrak{h}} \Rightarrow \rho(\mathfrak{h}) \subset$$

$$\subset \left\{ \begin{pmatrix} 0 & * & * \\ & & * \\ & & 0 \end{pmatrix} \right\} \subseteq \mathfrak{gl}(n,\mathbb{C})$$

$$\Rightarrow \pi(H/D) \leq \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \right\} =: L . \quad \blacksquare$$

Nilpotency

$\mathfrak{g}$ Lie algebra.

$$C^1(\mathfrak{g}) := [\mathfrak{g},\mathfrak{g}] \qquad (= \mathfrak{g}^{(1)})$$

$$C^{n+1}(\mathfrak{g}) := [\mathfrak{g}, C^n(\mathfrak{g})] =$$

$$= \mathrm{ad}(\mathfrak{g})(C^n(\mathfrak{g})) = \cdots$$

$$\cdots = \mathrm{ad}(\mathfrak{g})^n(\mathfrak{g})$$

Definition $\mathfrak{g}$ Lie algebra. The central series of $\mathfrak{g}$ is

$$\mathfrak{g} \supset C^1(\mathfrak{g}) \supset \cdots \supset C^n(\mathfrak{g}) \supset \cdots$$

We say that $\mathfrak{g}$ is nilpotent if $\exists n > 0$ s.t. $C^n(\mathfrak{g}) = \{0\}$.

Remark

(1) $\mathfrak{g}^{(1)} = C^1(\mathfrak{g})$

$$\mathfrak{g}^{(2)} = [\mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}] = [C^1(\mathfrak{g}), C^1(\mathfrak{g})]$$

$$\subset [\mathfrak{g}, C^1(\mathfrak{g})] = C^2(\mathfrak{g})$$

$$\vdots$$

$$\mathfrak{g}^{(n)} \subset C^n(\mathfrak{g}) \Rightarrow$$

every nilpotent Lie algebra is solvable. The converse is not true; for example

$$\mathfrak{g} = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in \mathfrak{sl}(2,\mathbb{R}) \right\}$$

$$\mathfrak{g}^{(1)} = \left\{ \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix} \in \mathfrak{sl}(2,\mathbb{R}) \right\}$$

but $\mathfrak{g}^{(2)} = \{0\}$ while

$$C^2(\mathfrak{g}) = [\mathfrak{g}, C^1(\mathfrak{g})] = C^1(\mathfrak{g})$$

(2) Each $C^j(\mathfrak{g})$ is a characteristic ideal and $C^j(\mathfrak{g})/C^{j+1}(\mathfrak{g})$ is Abelian. In fact

$$\left[C^j(\mathfrak{g})/C^{j+1}(\mathfrak{g}), C^j(\mathfrak{g})/C^{j+1}(\mathfrak{g}) \right] = 0$$

since $[C^j(\mathfrak{g}), C^j(\mathfrak{g})] \subset$

$$\subset [\mathfrak{g}, C^j(\mathfrak{g})] = C^{j+1}(\mathfrak{g}) .$$

What is more important is that

$$C^j(\mathfrak{g})/C^{j+1}(\mathfrak{g}) \subset Z\left(\mathfrak{g}/C^{j+1}(\mathfrak{g})\right)$$

$$\left(\text{that is } \left[C^j(\mathfrak{g})/C^{j+1}(\mathfrak{g}), \mathfrak{g}/C^{j+1}(\mathfrak{g}) \right] = \right.$$

$$\left. = 0 \text{ since } [C^j(\mathfrak{g}), \mathfrak{g}] = C^{j+1}(\mathfrak{g}) \right)$$

In particular if $C^{n+1}(\mathfrak{g}) = \{0\}$

$\Rightarrow$ $\boxed{C^n(\mathfrak{g}) \subset Z(\mathfrak{g})}$.

Solvable $\Rightarrow$ the last non-zero ideal $\mathfrak{g}^{(n)}$ in the derived series is Abelian

Nilpotent $\Rightarrow$ the last non-zero ideal $C^n(\mathfrak{g})$ in the central series is central.

So nilpotent Lie algebras must have a non-trivial center.

Ex. $\mathfrak{g} = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in \mathfrak{sl}(2, \mathbb{R}) \right\}$ not nilpotent and in fact $Z(\mathfrak{g}) = \{0\}$.