Exercise Class Lie Groups Dec 3^{rd}

Exercise 1.(Discrete Subgroups of $\mathbb{R}^n$):

Let $D < \mathbb{R}^n$ be a discrete subgroup. Show that there are $x_1, \dots, x_k \in D$ such that

- $x_1, \dots, x_k$ are linearly independent over $\mathbb{R}$, and
- $D = \mathbb{Z}x_1 \oplus \cdots \oplus \mathbb{Z}x_k$, i.e. $x_1, \dots, x_k$ generate D as a $\mathbb{Z}$-submodule of $\mathbb{R}^n$.

Relevance: (see lecture Nov 24^{th})

Any connected abelian Lie group is isomorphic to $\mathbb{R}^k \times \mathbb{T}^\ell$, $k, \ell \geq 0$.

Solution: Via induction on n ($= \dim \mathbb{R}^n$)

$n=1$: Take $\{0\} \neq D < \mathbb{R}$ discrete subgp.

Let $x_1 \in D$ with $|x_1| = \min\{|x| \mid x \in D \setminus \{0\}\}$.
Then $D = \mathbb{Z} \cdot x_1$ (check)

$n-1 \to n$: Assume assertion holds for $\{0\} \neq D \leq \mathbb{R}^{n-1}$

As before, pick $x_1 \in D$ with $\|x_1\| = \min\{\|x\| \mid x \in D \setminus \{0\}\}$.
Consider quotient map $\pi: \mathbb{R}^n \longrightarrow \mathbb{R}^n / \mathbb{R} x_1 \cong \mathbb{R}^{n-1}$
Claim: $D' := \pi(D) \leq \mathbb{R}^{n-1}$ discrete
Pf: (see solution) $\square$

By induction hypothesis: $\exists\, x_2', \dots, x_k' \in D' \leq \mathbb{R}^{n-1}$
linearly indep. s.t. $D' = \mathbb{Z} x_2' \oplus \dots \oplus \mathbb{Z} x_k'$
Pick $x_i \in \pi^{-1}(x_i') \cap D$ $(i = 2, \dots, k)$.

Then $x_1, x_2, \dots, x_k$ are linearly indep. and generate D.
(see solution)

Indeed, let $y \in D$. Because $D' = \mathbb{Z} x_2' \oplus \dots \oplus \mathbb{Z} x_k'$
$\exists\, a_2, \dots, a_k \in \mathbb{Z}$:

$$D' \ni \pi(y) = a_2 \cdot \underbrace{x_2'}_{=\pi(x_2)} + \dots + a_k \cdot \underbrace{x_k'}_{=\pi(x_k)} = \pi(\underbrace{a_2 x_2 + \dots + a_k \cdot x_k}_{=y'}).$$

$\Longrightarrow$ $y - y' \in \ker(\pi) \cap D$.

<u>Claim</u>: $\ker(\pi) \cap D = \mathbb{Z} \cdot x_1$

Pf: "$\supseteq$":

"$\subseteq$": Suppose $\exists\, t \cdot x_1 \in D$ with $t \in \mathbb{R} \setminus \mathbb{Z}$

Then $w = (t - \lfloor t \rfloor) \cdot x_1 \in D \setminus \{0\}$ &

$$\|w\| = \underbrace{(t - \lfloor t \rfloor)}_{\in (0,1)} \cdot \|x_1\| < \|x_1\|$$ $\lightning$ minimality of x_1.

$\Longrightarrow$ $y - y' \in \mathbb{Z} \cdot x_1$.

$\Longrightarrow$ $\exists\, a_1 \in \mathbb{Z}$:

$$y = a_1 \cdot x_1 + y' = a_1 x_1 + a_2 x_2 + \dots + a_k \cdot x_k. \qquad \square$$

Exercise 2.(Covering maps of Lie Groups):

Let G be a Lie group, let H be a simply connected topological space and let $p : H \to G$ be a covering map.

a) Show that there is a unique Lie group structure on H such that p is a smooth group homomorphism and that the kernel of p is a discrete subgroup of G.

Solution:

Existence (idea): $p: H \to G$ a covering, hence it's a local homeomorphism. Use this to define smooth charts on H by "pulling-back" charts from G.

Uniqueness: Suppose there are two universal covering Lie groups $p_1: H_1 \to G$, $p_2: H_2 \to G$. Because both H_1, H_2 are simply connected, there are lifts

$$
\begin{tikzcd}
 & H_2 \arrow[d, "p_2"] \\
H_1 \arrow[ur, dashed, "\varphi_1"] \arrow[r, "p_1"'] & G
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & H_1 \arrow[d, "p_1"] \\
H_2 \arrow[ur, dashed, "\varphi_2"] \arrow[r, "p_2"'] & G
\end{tikzcd}
$$

$\varphi_1(e_1) = e_2$, $\varphi_2(e_2) = e_1$

By uniqueness, $\varphi_1 = \varphi_2^{-1}$ and H_1 & H_2 are isomorphic as Lie groups.

p is a covering map

$\implies$ $\ker(p) = p^{-1}(\{e\})$ is discrete. $\square$

b) Show that p is a local isomorphism of Lie groups and that dp is an isomorphism of Lie algebras when H is equipped with the Lie group structure from part a).

Sol: p is a homomorphism & local diffeom

$\Rightarrow dp : \mathfrak{h} \cong T_eH \to T_eG = \mathfrak{g}$ is an isomorphism.

$\xRightarrow[\text{lecture}]{\text{Lemma}}$ p is a local isomorphism. $\square$

c) Let H, G be arbitrary Lie groups and let G be connected. Further, let $\varphi : H \to G$ be a Lie group homomorphism. Show that φ is a covering map if and only if $d\varphi : \mathfrak{h} \to \mathfrak{g}$ is an isomorphism.

Sol: "$\Rightarrow$": b) ✓

"$\Leftarrow$": $d\varphi : \mathfrak{h} \to \mathfrak{g}$ isom. $\xRightarrow{\text{IFT}}$ $\exists\, V \subset H,\ \mathcal{U} \subset G$:

$\varphi : \underset{e}{\underset{\Psi}{V}} \xrightarrow{\sim} \underset{e}{\underset{\Psi}{\mathcal{U}}}$ diffeom.

G is connected $\Rightarrow \langle \mathcal{U} \rangle_G = G \Rightarrow \varphi$ is surjective.

Evenly covered: Choose $e \in W = W^{-1} \subseteq V$ & $W^2 \subseteq V$.

Set $\mathcal{U}' = \varphi(W)$.

disjoint union

Claim: $\varphi^{-1}(\mathcal{U}') = \bigsqcup_{h \in \ker(\varphi)} W \cdot h$

Pf: Clearly, $\varphi^{-1}(\mathcal{U}') = \bigcup_{h \in \ker(\varphi)} W \cdot h$. (see solution)

$Wh \cap Wh' \neq \emptyset$ then $\exists\, w, w' \in W$ s.t.

$$wh = w'h' \Rightarrow h(h')^{-1} = w^{-1}w' \in W^{-1} \cdot W \subseteq V$$

$$\varphi|_V \text{ is injective} \Rightarrow h(h')^{-1} = e \Rightarrow h = h'$$

$\square$

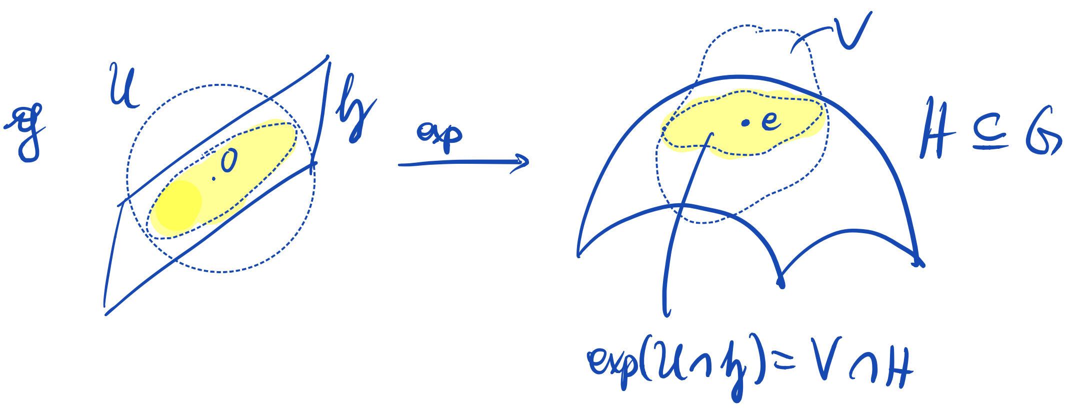
Exercise 3.(Abstract Subgroups as Lie Subgroups):

Let H be an abstract subgroup of a Lie group G and let $\mathfrak{h}$ be a subspace of the Lie algebra $\mathfrak{g}$ of G. Further let $U \subseteq \mathfrak{g}$ be an open neighborhood of $0 \in \mathfrak{g}$ and let $V \subseteq G$ be an open neighborhood of $e \in G$ such that the exponential map $\exp : U \to V$ is a diffeomorphism satisfying $\exp(U \cap \mathfrak{h}) = V \cap H$. Show that the following statements hold:

a) H is a Lie subgroup of G with the induced relative topology;

b) $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$;

c) $\mathfrak{h}$ is the Lie algebra of H.



Solution: $\exp|_{\mathcal{U}}^{-1} : V \longrightarrow \mathcal{U}$ gives a (local) slice chart at e. We can translate via left-transl. $L_h : G \to G,\ g \mapsto hg$ to obtain slice charts about every point $h \in H$.
Thus, $H \subseteq G$ is an embedded submanifold.

Because multiplication and inversion are smooth maps on G and restrict to H, they are smooth maps on H.

$\Longrightarrow$ $H \leq G$ is an embedded Lie subgroup.

$\square$

<u>b) & c)</u>: Denote by $\iota : H \hookrightarrow G$ the embedding.
Let $\mathfrak{b} \subseteq \mathfrak{g}$ be a complementary subsp. to $\mathfrak{h}$, i.e. $\mathfrak{h} \oplus \mathfrak{b} = \mathfrak{g}$. We get:

$$\begin{tikzcd} \mathrm{Lie}(H) \arrow[r, "d\iota"] \arrow[d, "\exp"] \arrow[dr, phantom, "//"] & \mathfrak{h} \oplus \mathfrak{b} = \mathfrak{g} \arrow[d, "\exp"] \\ H \arrow[r, "\iota"] & G \end{tikzcd}$$

By defn., $d\iota : \mathrm{Lie}(H) \to \mathfrak{h}$ is an isom. of vector spaces.
Because ι is a Lie group hom.
$d\iota : \mathrm{Lie}(H) \to \mathfrak{h} \subseteq \mathfrak{g}$ is a Lie algebra hom..

$\square$

Exercise 4.(Lie Group homomorphisms and their differentials):

Let G be a connected Lie group, let H be a Lie group and let $\varphi, \psi\colon G \to H$ be Lie group homomorphisms.

Show that $\varphi = \psi$ if and only if $d\varphi = d\psi$.

Sol: If $\varphi = \psi \Rightarrow d\varphi = d\psi$. ✓

$\Leftarrow$: Suppose $d\varphi = d\psi$. Consider

$$A = \{ g \in G \mid \varphi(g) = \psi(g) \}.$$

WTS: $A = G$.

Note: $e \in A$ & A is closed

Indeed, $\Delta_H = \{ (h,h) \in H \times H \} \subseteq H \times H$ is closed

s.t. $A = (\varphi \times \psi)^{-1}(\Delta_H)$ is closed.

If A is open then $A = G$ by connectedness of G.

Recall, that there are open nbhds $0 \in \mathcal{U} \subseteq \mathfrak{g}$, $e \in V \subseteq G$ s.t. $\exp|: \mathcal{U} \xrightarrow{\sim} V$ is a diffeo.

Pick $g_0 \in A$. Let $g = g_0 v \in g_0 V$ and $X \in \mathcal{U}$ s.t. $\exp(X) = v$.

Then:

$$\varphi(g) = \varphi(g_0)\varphi(v) = \varphi(g_0)\varphi(\exp(X)) = \varphi(g_0)\exp(d\varphi(X))$$

$$= \psi(g_0)\exp(d\psi(X)) = \ldots = \psi(g).$$

$\Rightarrow g_0 V \subseteq A \Rightarrow A$ is open. $\square$

Exercise 5.(Surjectivity of the Matrix Exponential):

Let $\mathrm{Exp} : \mathfrak{gl}(n,\mathbb{R}) \cong \mathbb{R}^{n\times n} \to \mathrm{GL}(n,\mathbb{R})$ be the matrix exponential map given by the power series

$$\mathrm{Exp}(X) := \sum_{n=0}^{\infty} \frac{X^n}{n!}.$$

Consider the Lie subgroup of upper triangular matrices $N(n) < \mathrm{GL}(n,\mathbb{R})$ with its Lie algebra $\mathfrak{n}(n) < \mathfrak{gl}(n,\mathbb{R})$ of strictly upper triangular matrices; cf. exercise sheet 4 problem 3.

Show that $\mathrm{Exp}|_{\mathfrak{n}(n)} : \mathfrak{n}(n) \to N(n)$ is surjective.

Hint: Consider the partially defined matrix logarithm $\mathrm{Log} : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ given by

$$\mathrm{Log}(I+A) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{A^n}{n}.$$

Try to give answers to the following questions and then conclude:

What is its radius of convergence r about I? Why is it a right-inverse of Exp on the ball $B_r(I)$ of radius r about I? Why is there no problem for matrices that are in $N(n)$ but not in $B_r(I)$?

In order to answer the last question prove that $A^n = 0$ for all $A \in \mathfrak{n}(n)$.

Solution: As in the complex case

$$r = \lim_{n\to\infty} \left| \frac{(-1)^{n-1}}{n} \cdot \frac{n+1}{(-1)^n} \right| = 1$$

$\Rightarrow$ $\mathrm{Log}(I+A)$ converges absolutely for all $A \in \mathbb{R}^{n\times n}$ with $\|A\| < 1$.

Claim: $\mathrm{Exp}(\mathrm{Log}(I+A)) = I + A \quad \forall \|A\| < 1.$

Pf: Via comparison with the complex case: There's a procedure to compute the coefficients of the series

$$\mathrm{Exp}(\mathrm{Log}(I+A)) = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\sum_{n=1}^{\infty} (-1)^{n-1} \frac{A^n}{n} \right)^k$$

$$= \sum_{k=0}^{\infty} \underline{\underline{d_k}} \cdot A^k.$$

and they are the $\underline{\underline{\text{same}}}$ as in the complex case:

$$\exp(\log(1+z)) = 1+z \qquad (\text{for } |z|<1)$$

$\Rightarrow \quad d_0 = 1, \quad d_1 = 1, \quad 0 = d_2 = d_3 = \dots$

$\Rightarrow \quad \mathrm{Exp}(\mathrm{Log}(I+A)) = I+A.$

$\square$

$g \in N$ is of the form

$$g = \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} = I + A^{\in \mathfrak{n}} \qquad \left(\begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix}^n = 0 \right)$$

Moreover, $A^n = 0$, s.t.

$$\mathrm{Log}(I+A) = \sum_{k=0}^{n-1} \frac{1}{k} (-1)^{k-1} A^k$$ is a polynomial and is defined $\forall A \in \mathfrak{n}$.

$\underline{\text{Claim:}}$ $\mathrm{Exp}(\mathrm{Log}(I+A)) = I+A$ for all $A \in \mathfrak{n}$

Pf: Let $A \in \mathfrak{n} \setminus \{0\}$. Then $\forall\, t \in (-\|A\|^{-1}, \|A\|^{-1})$
$(\Rightarrow \|t \cdot A\| < 1)$

$$\mathrm{Exp}(\mathrm{Log}(I + t \cdot A)) = I + t \cdot A.$$

Both sides are analytic functions in t and coincide on the open set $(-\|A\|^{-1}, \|A\|^{-1})$.

$\Rightarrow$ They must coincide on all of $\mathbb{R}$.
In particular, for $t = 1$.

$$\mathrm{Exp}(\mathrm{Log}(I + A)) = I + A.$$

$\square$

Exercise 6.(Multiplication and exp):

Let G be a Lie group with Lie algebra $\mathfrak{g}$. Show that for all $X, Y \in \mathfrak{g}$ and small enough $t \in \mathbb{R}$

$$\exp(tX)\exp(tY) = \exp(t(X+Y) + O(t^2))$$

where $O(t^2)$ is a differentiable $\mathfrak{g}$-valued function such that $\frac{O(t^2)}{t^2}$ is bounded as $t \to 0$.

Solution: Let $X, Y \in \mathfrak{g}$, $\mathcal{U} \subseteq \mathfrak{g}$, $V \subseteq G$
s.t. $\exp|_{\mathcal{U}} : \mathcal{U} \xrightarrow{\sim} V$

Let $\varepsilon > 0$ s.t. $\exp(t \cdot X) \cdot \exp(t \cdot Y) \in V \quad \forall\, |t| < \varepsilon$.
$\Rightarrow$ There is $Z : (-\varepsilon, \varepsilon) \longrightarrow \mathfrak{g}$ s.t.
$\exp(Z(t)) = \exp(t \cdot X) \cdot \exp(t \cdot Y)$.

Taylor: $\quad Z(t) = Z(0) + t \cdot Z'(0) + O(t^2)$.

$$\exp(Z(0)) = \underbrace{\exp(0 \cdot X)}_{=e} \cdot \underbrace{\exp(0 \cdot Y)}_{=e} = e$$

$$\Longrightarrow \quad Z(0) = 0.$$

For $f \in C^\infty(G)$:

$$\frac{d}{dt}\Big|_{t=0} f(\overbrace{\exp(t \cdot X) \cdot \exp(t \cdot Y)}^{=\exp(Z(t))}) \qquad (t \mapsto (t,t))$$

$$= \frac{d}{dt}\Big|_{t=0} f(\exp(t \cdot X) \cdot \cancel{\exp(0 \cdot Y)}^{e}) + \frac{d}{dt}\Big|_{t=0} f(\cancel{\exp(0 \cdot X)}^{e} \exp(t \cdot Y))$$

$$= X \cdot f + Y \cdot f$$

$$= \frac{d}{dt}\Big|_{t=e} f(\exp(Z(t))) = Z'(0) \cdot f$$

$$\Longrightarrow \quad Z'(0) = X + Y.$$

$$\Longrightarrow \quad \exp(t \cdot X) \exp(t \cdot Y) = \exp(Z(t))$$

$$= \exp(t \cdot (X+Y) + O(t^2)).$$

$\square$