09 December 2020

- $\mathfrak{g}$ nilpotent if $C^n(\mathfrak{g}) = \{0\}$,
  $C^{j+1}(\mathfrak{g}) := [\mathfrak{g}, C^j(\mathfrak{g})]$,
  $C^1(\mathfrak{g}) := [\mathfrak{g},\mathfrak{g}]$
- nilpotent $\Rightarrow$ solvable
  $C^i(\mathfrak{g}) \supset \mathfrak{g}^{(i)}$
- If $C^n(\mathfrak{g}) = \{0\} \Rightarrow C^{n-1}(\mathfrak{g}) \subset Z(\mathfrak{g})$
  That is a nilpotent Lie alg. must have non-trivial center.

Proposition TFAE:

(1) $\mathfrak{g}$ nilpotent

(2) $\exists$ chain of subalg.
$\mathfrak{g} \supset \mathfrak{g}_1 \supset \dots \supset \mathfrak{g}_n = \{0\}$ s.t.
a) $\mathfrak{g}_{i+1}$ ideal in $\mathfrak{g}_i$
b) $[\mathfrak{g},\mathfrak{g}_i] \subset \mathfrak{g}_{i+1}$

(3) $\exists$ $p \in \mathbb{N}$ s.t.
$\mathrm{ad}(x_1) \circ \dots \circ \mathrm{ad}(x_p) = 0$
$\forall\, x_1, \dots, x_p \in \mathfrak{g}$

Pf (1) $\Rightarrow$ (2) obvious

(2) $\Rightarrow$ (1) By induction
Set $\mathfrak{g}_0 := \mathfrak{g}$
$C^1(\mathfrak{g}) = [\mathfrak{g},\mathfrak{g}_0] \overset{b)}{\subset} \mathfrak{g}_1$
Assume $C^j(\mathfrak{g}) \subset \mathfrak{g}_j$;
$C^{j+1}(\mathfrak{g}) = [\mathfrak{g}, C^j(\mathfrak{g})] \subset$
$\subset [\mathfrak{g},\mathfrak{g}_j] \subset \mathfrak{g}_{j+1}$

(1) $\Leftrightarrow$ (3) $C^k(\mathfrak{g})$ is generated by elem. of the form
$\mathrm{ad}(x_1) \circ \dots \circ \mathrm{ad}(x_k)$ ■

Recall $\mathfrak{g}$ solvable $\Leftrightarrow$ $\mathfrak{h}$, $\mathfrak{g}/\mathfrak{h}$ solv. ($\mathfrak{h}$ ideal)

$\mathfrak{g}$ nilp $\not\Leftrightarrow$ $\mathfrak{h}$, $\mathfrak{g}/\mathfrak{h}$ nilp.

Prop. $\mathfrak{h} \subset \mathfrak{g}$ ideal.
(1) $\mathfrak{g}$ nilp $\Rightarrow$ $\mathfrak{h}$, $\mathfrak{g}/\mathfrak{h}$ nilp.
(2) $\mathfrak{g}/\mathfrak{h}$ nilp, $\mathfrak{h} \subset Z(\mathfrak{g}) \Rightarrow \mathfrak{g}$ nilp.

Pf (1) obvious

(2) $\mathfrak{g}/\mathfrak{h}$ nilpotent $\Rightarrow \exists$ $\mathfrak{h}_j$ subalg. (as in Prop.)
$\mathfrak{g}/\mathfrak{h} \supset \mathfrak{h}_1 \supset \dots \supset \mathfrak{h}_n = \{0\}$
$p: \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ projection (Lie alg. homo.)
$\mathfrak{g} \supset p^{-1}(\mathfrak{h}_1) \supset \dots \supset p^{-1}(\mathfrak{h}_n) = \mathfrak{h} \supset \mathfrak{h}'_{n+1} = (0)$
satisfy the properties of the prop. ■

Defn. G conn. Lie gp is nilp. if $\mathrm{Lie}(G)$ is nilp.

Prop. TFAE:
a) G nilpotent
b) $\exists$ sequence of closed connected normal subgps $G_i \triangleleft G$ s.t.
i) $[G, G_i] < G_{i+1}$
ii) $G \supset G_1 \supset \dots \supset G_n = \{e\}$

c) $\exists$ sequence of closed connected normal subgroups $G_i \triangleleft G$ s.t.
i) $G_i/G_{i+1} < Z(G/G_{i+1})$
ii) $G \supset G_1 \supset \dots \supset G_n = \{e\}$,

Pf like for solvable. Exercise. ■

Proposition $\mathfrak{g}$ solvable $\Leftrightarrow$
$\Leftrightarrow$ $[\mathfrak{g},\mathfrak{g}]$ nilpotent.

Pf ($\Leftarrow$) $[\mathfrak{g},\mathfrak{g}]$ nilp. $\Rightarrow$
$\Rightarrow$ $[\mathfrak{g},\mathfrak{g}]$ solvable. Also
$\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$ solvable $\Rightarrow \mathfrak{g}$ solv.

($\Rightarrow$) (i) $\mathfrak{g} \subset \mathfrak{gl}(V)$ solvable, V $\mathbb{C}$-vector space. Lie's thm $\Rightarrow$
$\Rightarrow$ $\mathfrak{g}$ u.t. $\Rightarrow$ $[\mathfrak{g},\mathfrak{g}]$ s.u.t. $\Rightarrow$
$\Rightarrow$ $[\mathfrak{g},\mathfrak{g}]$ nilpotent.

(ii) $\mathfrak{g}$ complex solvable Lie algebra (not nec. $\mathfrak{g} \subset \mathfrak{gl}(V)$)

$\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}) \Rightarrow$

(i) $\Rightarrow [\mathrm{ad}(\mathfrak{g}), \mathrm{ad}(\mathfrak{g})] = \mathrm{ad}\,[\mathfrak{g},\mathfrak{g}]$ is nilpotent. want $[\mathfrak{g},\mathfrak{g}]$ nilp.!

$$\mathrm{ad}_{\mathfrak{g}}([\mathfrak{g},\mathfrak{g}]) = [\mathfrak{g},\mathfrak{g}] / Z_{\mathfrak{g}}[\mathfrak{g},\mathfrak{g}]$$

But $Z_{\mathfrak{g}}[\mathfrak{g},\mathfrak{g}] = \ker(\mathrm{ad}|_{[\mathfrak{g},\mathfrak{g}]}) < Z(\mathfrak{g}) \Rightarrow [\mathfrak{g},\mathfrak{g}]$ nilpotent.

(iii) $\mathfrak{g}$ real solvable Lie alg. $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} + i\mathfrak{g}$ solvable $\Rightarrow [\mathfrak{g}^{\mathbb{C}}, \mathfrak{g}^{\mathbb{C}}]$ nilpotent and $[\mathfrak{g},\mathfrak{g}] \subset [\mathfrak{g}^{\mathbb{C}}, \mathfrak{g}^{\mathbb{C}}]$ nilpotent. ∎

Engel's thm Let $\mathfrak{g} \subset \mathfrak{gl}(V)$ a Lie algebra, V k-vector space over any field k, and we suppose that $X^n = 0$ for some $n \in \mathbb{N}$, $\forall X \in \mathfrak{g}$. Then $\exists$ basis of V

s.t. $\mathfrak{g}$ is s.u.t.

Rk $\mathfrak{g}$ s.u.t. $\Rightarrow \mathfrak{g}$ nilp.

$\mathfrak{g}$ nilp $\not\Rightarrow \mathfrak{g}$ s.u.t. (for ex. $\mathfrak{g} = \left\{\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}\right\}$.)

Corollary $\mathfrak{g}$ nilpotent $\Leftrightarrow \mathrm{ad}(\mathfrak{g})$ s.u.t.

Pf ($\Rightarrow$) $\mathfrak{g}$ nilpotent $\Rightarrow \mathrm{ad}(x)^n = 0$ $\forall x \in \mathfrak{g}$. Then apply Engel's thm.

($\Leftarrow$) $\mathrm{ad}(\mathfrak{g})$ s.u.t. $\Rightarrow \mathrm{ad}(\mathfrak{g})$ nilp.

$$\underbrace{\mathrm{ad}(\mathfrak{g})}_{\text{nilp}} = \mathfrak{g}/Z(\mathfrak{g}) \Rightarrow \mathfrak{g} \text{ nilp.} \ \blacksquare$$

To prove Engel's thm it will be enough to show:

Thm If $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ repr. s.t. $\rho(X)$ nilp. $\forall X \in \mathfrak{g} \Rightarrow \rho(\mathfrak{g})$ has a common null vector, that is

$\exists\, v \in V \setminus \{0\}$ s.t. $\rho(X)v = 0$ $\forall X \in \mathfrak{g}$.

In fact If we can prove this, let V_0 be the space of null vectors, $\Rightarrow$

$$\Rightarrow \rho(\mathfrak{g}) < \begin{array}{cc} & \overbrace{\quad}^{V_0} \\ V_0 \{ & \left(\begin{array}{c|c} 0 & * \\ 0 & * \end{array}\right) \end{array}$$

$$\Rightarrow \rho_0 : \mathfrak{g} \to \mathfrak{gl}(V/V_0)$$

Lemma $X \in \mathfrak{g} \subset \mathfrak{gl}(V)$ nilpotent $\Rightarrow$

$\Rightarrow \mathrm{ad}(X)$ nilpotent.

Pf Exercise $\left[\begin{array}{l} R_X, L_X \in \mathrm{End}(\mathfrak{g}) \\ \mathrm{ad}(X)Y = L_X Y - R_X Y \\ L_X, R_X \text{ nilp.} \end{array}\right]$

Pf By induction on $\dim \mathfrak{g}$.

$\dim \mathfrak{g} = 1$: $\mathfrak{g} = kX$, $\rho(X)^n = 0$

Assume n is the smallest

s.t. $\rho(X)^n = 0$. If $v \in V \setminus \{0\}$ is a null vector $\Rightarrow \rho(X)^n v = 0$ but $\rho(X)^{n-1} v \neq 0$. Hence $\rho(X)^{n-1} v$ is a null vector for $\mathfrak{g}$.

$\dim \mathfrak{g} > 1$ Assume assertion is true $\forall\, \mathfrak{h}$ with $\dim \mathfrak{h} < \dim \mathfrak{g}$.

Assume also that ρ is faithful: if not $\rho(\mathfrak{g})/\ker \rho$ has dim. smaller than $\mathfrak{g}$. Idea of proof:

1) $\mathfrak{g} = kX_0 \oplus \mathfrak{h}$, $\mathfrak{h}$ ideal
2) look for a null vector of X_0 in the space of null vectors of $\mathfrak{h}$.

(1) $\mathfrak{h}$ maximal proper subalgebra

$\mathrm{ad}_{\mathfrak{g}}(X) : \mathfrak{g} \to \mathfrak{g}$. If $X \in \mathfrak{h} \Rightarrow$

$\Rightarrow \mathrm{ad}_{\mathfrak{g}}(X) : \mathfrak{h} \to \mathfrak{h} \Rightarrow$

$$\mathrm{ad}_{\mathfrak{g}} : \mathfrak{h} \to \mathfrak{gl}(\mathfrak{g}/\mathfrak{h})$$

$\Rightarrow$ By ind. hyp. $\exists\, X_0 \in \mathfrak{g}/\mathfrak{h}$ and $X_0 \neq 0$ s.t. $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{h}) X_0 = 0$ in $\mathfrak{g}/\mathfrak{h}$

$\Rightarrow \underline{\mathrm{ad}_{\mathfrak{g}}(\mathfrak{h}) X_0 \overset{(*)}{\in} \mathfrak{h}}$.

Also $X_0 \in \mathfrak{g}/\mathfrak{h}$, $X_0 \neq 0 \Rightarrow \underline{X_0 \notin \mathfrak{h}}$

we'll use this to show that $kX_0 \oplus \mathfrak{h}$ is a subalgebra which because of will be larger than $\mathfrak{h}$ (which was maximal) $\Rightarrow kX_0 \oplus \mathfrak{h} = \mathfrak{g}$.

$$[kX_0 \oplus \mathfrak{h}, kX_0 \oplus \mathfrak{h}] =$$

$$= \underbrace{[kX_0, kX_0]}_{=0} + \underbrace{[kX_0, \mathfrak{h}]}_{\overset{(*)}{\subset}\, \mathfrak{h}} + \underbrace{[\mathfrak{h},\mathfrak{h}]}_{\subset\, \mathfrak{h}}$$

$\Rightarrow kX_0 \oplus \mathfrak{h}$ is a subalgebra.

$\Rightarrow \mathfrak{h}$ is an ideal.

(2) W = space of null vectors of $\mathfrak{h}$. Want to see that it is X_0-inv.

$w \in W$, $\forall X \in \mathfrak{h}$, $\rho(X) w = 0$

Is $\rho(X)\rho(X_0) w \overset{?}{=} 0$ $\forall X \in \mathfrak{h}$?

$$\rho(X)\rho(X_0) w = \rho(\underbrace{[X, X_0]}_{\in\, \mathfrak{h}\text{ (ideal)}}) w + \rho(X_0)\underbrace{\rho(X) w}_{=0} = 0$$

$\blacksquare$

Corollary $\mathfrak{g}$ nilpotent, $\mathfrak{h}$ max. subalgebra $\Rightarrow$ $\mathfrak{h}$ ideal, $[\mathfrak{g},\mathfrak{g}] \subset \mathfrak{h}$ and $\mathfrak{h}$ has codim 1.

In fact we need it if $\forall X \in \mathfrak{g}$ is nilpotent. $\Rightarrow$ $\boxed{\mathrm{ad}(X) \text{ nilp.}}$ $\forall X \in \mathfrak{g}$. This is all was needed in the proof.

So ass. that $\mathfrak{g}$ nilp is enough.

ad(X) nilpotent $\forall X \in \mathfrak{g}$

$\Uparrow$ $\Uparrow$

X nilp $\forall X \in \mathfrak{g}$ $\mathfrak{g}$ nilp.

Cor.

Killing form

Recall V k-vector space, $k = \mathbb{R}, \mathbb{C}$, $A \in \mathrm{End}(V)$

$\mathrm{tr}(A) = \sum \lambda_i$ λ_i e.v. of A

1) $\mathrm{tr}(XAX^{-1}) = \mathrm{tr}(A)$, $X \in GL(V)$

2) $\mathrm{tr}(AB) = \mathrm{tr}(BA)$, $A, B \in \mathrm{End}(V)$

3) Choose a basis of V my $A = (a_{ij})_{ij} \Rightarrow$

$$\Rightarrow \mathrm{tr}(A) = \sum_{i=1}^{n} a_{ii}.$$

Defn (1) The trace form is the bilinear form

$$B : k^{n\times n} \times k^{n \times n} \longrightarrow k$$

defn as $B(X,Y) := \mathrm{tr}(XY)$

(2) Let $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ repr. The trace form of ρ is

$$B_\rho : \mathfrak{g} \times \mathfrak{g} \longrightarrow k$$

$$(X,Y) \longmapsto B(\rho(X)\rho(Y)) = \mathrm{tr}(\rho(X)\rho(Y))$$

(3) The Killing form is the trace form of ad.

$$B_{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g} \longrightarrow k$$

$$(X,Y) \longmapsto \mathrm{tr}(\mathrm{ad}(X)\mathrm{ad}(Y))$$

Defn. V k-vector space, $f : V \times V \to k$ bilinear form, $G \leq GL(V)$, $\mathfrak{g} \subseteq \mathfrak{gl}(V)$:

(1) f is G-inv. if

$$f(AX, AY) = f(X,Y) \quad \forall X, Y \in V, \ \forall A \in G.$$

(2) f is $\mathfrak{g}$-inv. if $\forall X, Y \in V$, $D \in \mathfrak{g}$,

$$f(DX, Y) + f(X, DY) = f(X,Y)$$

Proposition $f: V\times V \to \mathbb{R}$ bil. form, $A \in End(V)$. Then

f is A-inv. ($f(AX,Y)+f(X,AY)=f(X,Y)$) $\iff$

$\iff f(\exp(tA)X, \exp(tA)Y) = f(X,Y)$ $\forall t\in\mathbb{R}$ $\forall X,Y\in V$.

Pf ($\Leftarrow$) Differentiate

($\Rightarrow$) $\varphi(t) := f(\exp(tA)X, \exp(tA)Y)$

$\psi(t) := f(X,Y)$.

Claim that both are solutions of $\frac{dz}{dt}=0$ $z(0)=f(X,Y)$.

Obviously $\psi(t)$ is.

Diff. $\varphi(t)$ and use the hyp. to show that also $\varphi(t)$ is a solution. By uniqueness $=$ ∎.

Corollary G Lie gp, $\mathfrak{g} = Lie(G)$, $f: V\times V\to\mathbb{R}$ bil. form. f is G-inv. $\iff$ f is $\mathfrak{g}$-inv.

Prop. $\rho: \mathfrak{g}\to \mathfrak{gl}(n,\mathbb{R})$ Lie alg. repr. $\Rightarrow$ B_ρ is $ad(\mathfrak{g})$-inv. In particular the Killing form is $ad(\mathfrak{g})$-invariant.

Pf Straightforwar verif. using $tr(XY) = tr(YX)$. ∎

Cartan's criterion for solvability

of Lie alg. with Killing form $B_{\mathfrak{g}}$.

$\mathfrak{g}$ solvable $\iff$ $B_{\mathfrak{g}}|_{\mathfrak{g}^{(1)}\times\mathfrak{g}^{(1)}} \equiv 0$

This follows from the following two facts:

Lemma $\mathfrak{h}\subset\mathfrak{g}$ ideal $\Rightarrow$

$\Rightarrow$ $B_{\mathfrak{h}} = B_{\mathfrak{g}}|_{\mathfrak{h}\times\mathfrak{h}}$

(Rk: $ad_{\mathfrak{g}}(X)|_{\mathfrak{h}} = ad_{\mathfrak{h}}(X)$ $\forall X\in\mathfrak{h}$ if $\mathfrak{h}$ is a subalgebra. For the Killing form, need $\mathfrak{h}$ to be an ideal).

Pf $\mathfrak{g} = \mathfrak{h}\oplus V$. If $X\in\mathfrak{h}$

- $ad_{\mathfrak{g}}(X)Y = [X,Y]\in\mathfrak{h}$ if $Y\in\mathfrak{h}$ ($\mathfrak{h}$ subalg.)

- $\forall Y\in V$, $ad_{\mathfrak{g}}(X)Y = [X,Y]\in\mathfrak{h}$ ($\mathfrak{h}$ ideal)

$\Rightarrow$ $ad_{\mathfrak{g}}(X) = \left(\begin{array}{c|c} ad_{\mathfrak{h}}(X) & * \\ \hline 0 & 0\end{array}\right)$

$\Rightarrow tr(ad_{\mathfrak{g}}(X)ad_{\mathfrak{g}}(Y)) =$

$= tr(ad_{\mathfrak{h}}(X)ad_{\mathfrak{h}}(Y))$. ∎

Thm Let $\mathfrak{g}\subset\mathfrak{gl}(V)$. If $tr(XY)=0$ $\forall X,Y\in\mathfrak{g}$ $\Rightarrow$

$\Rightarrow$ $\mathfrak{g}^{(1)}$ is s.u.t. In particular $\mathfrak{g}^{(1)}$ is nilp. and $\mathfrak{g}$ is solvable.