16 December 2020

- semisimple Lie algebras
- Levi decomposition
- compact groups

Last time

Thm $\mathbb{K}=\mathbb{R},\mathbb{C}$, $\mathfrak{g}\subset \mathfrak{gl}(n,\mathbb{K})$ self-adjoint and $Z(\mathfrak{g})=\{0\} \Rightarrow$
$\Rightarrow B_{\mathfrak{g}}$ non-degenerate (hence $\mathfrak{g}$ is semisimple)

Lemma If $W\subset \mathbb{K}^n$ self-adjoint w.r.t. some inner product $\Rightarrow$ trace form restr. to W is non-deg.

Pf of thm $B_{\mathfrak{g}}: \mathfrak{g}\times\mathfrak{g}\to\mathbb{K}$

$$\mathfrak{g}\times\mathfrak{g} \xrightarrow{\mathrm{ad}_{\mathfrak{g}}\times\mathrm{ad}_{\mathfrak{g}}} \mathrm{ad}_{\mathfrak{g}}(\mathfrak{g})\times\mathrm{ad}_{\mathfrak{g}}(\mathfrak{g}) \xhookrightarrow{i} \mathfrak{gl}(\mathfrak{g})\times\mathfrak{gl}(\mathfrak{g}) \xrightarrow{\mathrm{tr}} \mathbb{K}$$

lemma $\Rightarrow$ If $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{g})$ is self-adj.
$\Rightarrow \mathrm{tr}|_{\mathrm{ad}_{\mathfrak{g}}(\mathfrak{g})}$ non-degenerate.

$Z(\mathfrak{g})=\{0\} \Rightarrow \mathfrak{g}\cong \mathrm{ad}_{\mathfrak{g}}(\mathfrak{g})$

Need to prove that
$\mathfrak{g}$ self-adj $\Rightarrow \mathrm{ad}_{\mathfrak{g}}(\mathfrak{g})$ self-adj.

Lemma $\langle\,,\,\rangle_t : \mathfrak{gl}(n,\mathbb{K})\times\mathfrak{gl}(n,\mathbb{K})\to\mathbb{K}$
$(X,Y)\mapsto \mathrm{tr}(XY^*)$

Y^* w.r.t. some inner pr. on $\mathbb{K}^{n\times n}\simeq \mathfrak{gl}(n,\mathbb{K})$.

$\mathrm{ad}_{\mathfrak{gl}(n,\mathbb{K})}: \mathfrak{gl}(n,\mathbb{K})\to\mathfrak{gl}(\mathfrak{gl}(n,\mathbb{K}))$

Then if $A\in\mathfrak{gl}(n,\mathbb{K})$

$$\mathrm{ad}_{\mathfrak{gl}(n,\mathbb{K})}(A^*) = \mathrm{ad}_{\mathfrak{gl}(n,\mathbb{K})}(A)^*$$

Pf We need to verify that $\forall X,Y\in\mathfrak{gl}(n,\mathbb{K}), \not\ni A$,

$$\langle \mathrm{ad}_{\mathfrak{gl}(n,\mathbb{K})}(A)X, Y\rangle_t \overset{?}{=}$$
$$\langle X, \mathrm{ad}_{\mathfrak{gl}(n,\mathbb{K})}(A^*)Y\rangle_t$$

In fact $\langle \mathrm{ad}_{\mathfrak{gl}(n,\mathbb{K})}(A)X, Y\rangle_t =$

$$= \langle [A,X],Y\rangle_t = \mathrm{tr}([A,X]Y^*) =$$

$$= \mathrm{tr}(AXY^*-XAY^*) = \mathrm{tr}(AXY^*) - \mathrm{tr}(XAY^*)$$

$$= \mathrm{tr}(XY^*A - XAY^*) = \mathrm{tr}(X[Y^*,A]) =$$

$$= \langle X, \mathrm{ad}_{\mathfrak{gl}(n,\mathbb{K})}(A^*)Y\rangle_t. \quad \blacksquare$$

Upshot If $\mathfrak{g}\subset\mathfrak{gl}(n,\mathbb{K})$ is self-adj
$\Rightarrow \mathrm{ad}_{\mathfrak{gl}(n,\mathbb{K})}(\mathfrak{g})$ self-adj.

But we wanted to show that
$\mathrm{ad}_{\mathfrak{g}}(\mathfrak{g})$.

But $\mathfrak{g}\subset\mathfrak{gl}(n,\mathbb{K})\simeq\mathbb{K}^{n\times n}$ subalgebra
$\mathfrak{g}=V$

$$\mathfrak{h}=\mathrm{ad}_{\mathfrak{gl}(n,\mathbb{K})}(\mathfrak{g})\subset\begin{pmatrix}\mathrm{ad}_{\mathfrak{g}}(\mathfrak{g}) & * \\ 0 & 0\end{pmatrix}$$
$m=n^2$

Lemma If $\mathfrak{h}\subset\mathfrak{gl}(m,\mathbb{K})$ is a self-adj. Lie algebra w.r.t. some inner prod. in $\mathbb{K}^m$ and $V\subset\mathbb{K}^m$ $\mathfrak{h}$-inv. subsp. $\Rightarrow \mathfrak{h}|_V$ self-adj.

$$\Rightarrow \mathrm{ad}_{\mathfrak{gl}(n,\mathbb{K})}(\mathfrak{g})\big|_{\mathfrak{g}} \text{ self-adj.}$$
$\quad\|$
$\mathrm{ad}_{\mathfrak{g}}(\mathfrak{g}) \Rightarrow$ done once we have proven the lemma.

Pf of lemma $\mathfrak{h}$ self-adj.
& V $\mathfrak{h}$-invariant $\Rightarrow V^\perp$ $\mathfrak{h}$-inv.
Thus if $H\in\mathfrak{h}$ we can write $H=H_V+H_{V^\perp}$ and
$H^*=H_V^*+H_{V^\perp}^*$ $\blacksquare$.

Alternative way to see non-degeneracy

Corollary V $\mathbb{C}$-vector space
If $\mathfrak{g}\subset\mathfrak{sl}(V)$ is irreducible & self-adjoint $\Rightarrow Z(\mathfrak{g})=\{0\}$ and hence $\mathfrak{g}$ is semisimple.

Here irreducible means as algebra of endom. of V, i.e. $\nexists$ non-trivial invariant subspaces in V.

Ex. $\mathfrak{sl}(V)$ & $\mathfrak{su}(n)$ are irreducible $\Rightarrow$ semisimple

Schur's lemma $\mathfrak{g}$ Lie alg. acting irred. on a cplx. v.s. V and $A: V \to V$ endom. that commutes with $\mathfrak{g}$ $\Rightarrow$ $A = \lambda I$, $\lambda \in \mathbb{C}$.

Pf λ e.value of A, consider $A - \lambda I$, which also commutes with $\mathfrak{g}$ $\Rightarrow$ $\ker(A - \lambda I)$ is a $\mathfrak{g}$-inv. subspace $\Rightarrow$

$\Rightarrow \ker(A - \lambda I) = V \Rightarrow$

$\Rightarrow A = \lambda I$ ∎

Corollary V $\mathbb{C}$-vector space. If $\mathfrak{g} \subset \mathfrak{sl}(V)$ is irreducible & self-adjoint $\Rightarrow Z(\mathfrak{g}) = \{0\}$ and hence $\mathfrak{g}$ is semisimple.

Pf of Corollary If $A \in \mathfrak{g} \subset \mathrm{End}(V)$ commutes with $\mathfrak{g}$ $\overset{\text{Schur}}{\Longrightarrow}$ $A = \lambda I$.

But $A \in \mathfrak{g} \subset \mathfrak{sl}(V) = \{\text{traceless matrices in } \mathfrak{gl}(V)\}$.

$\Rightarrow 0 = \operatorname{tr}(A) = n\lambda \Rightarrow \lambda = 0$ ∎

Proposition $\mathfrak{g} = \sum_{i \in I}^{\oplus} \mathfrak{g}_i$, $\mathfrak{g}_i$ simple ideals. Then any ideal of $\mathfrak{g}$ is of the form

$\sum_{i \in J}^{\oplus} \mathfrak{g}_i$, $J \subset I$.

Pf $\mathfrak{h} \subset \mathfrak{g}$ ideal and let J be the smallest subset s.t. $\mathfrak{h} \subset \sum_{j \in J}^{\oplus} \mathfrak{g}_j$. Let $j \in J$.

$\mathfrak{g}_j$ ideal $\Rightarrow [\mathfrak{h}, \mathfrak{g}_j] \subseteq \mathfrak{g}_j$

$\mathfrak{g}_j$ simple ideal $\Rightarrow [\mathfrak{h}, \mathfrak{g}_j] = \{0\}$ or $\mathfrak{g}_j$

Want to show that $[\mathfrak{h}, \mathfrak{g}_j] = \mathfrak{g}_j$.

If so $\Rightarrow \mathfrak{g}_j = [\mathfrak{h}, \mathfrak{g}_j] \subset \mathfrak{h} \Rightarrow$ ($\mathfrak{h}$ ideal)

$\Rightarrow \mathfrak{h} = \sum_{j \in J}^{\oplus} \mathfrak{g}_j$.

To show $[\mathfrak{h}, \mathfrak{g}_j] \neq \{0\}$.

Let $X = X_1 + \dots + X_n \in \mathfrak{h}$, $X_i \in \mathfrak{g}_i$ $\forall i \in J$, $n = |J|$.

We can assume $X_j \neq 0$ (otherwise contr. the min. of J).

If $[\mathfrak{h}, \mathfrak{g}_j] = \{0\} \Rightarrow$

$[X, \mathfrak{g}_j] = \{0\} \Rightarrow [X_j, \mathfrak{g}_j] = 0$

$\Rightarrow X_j \in Z(\mathfrak{g}_j) = \{0\}$. ↯ ∎

Corollary (1) Any s.s. Lie alg. has a finite no. of ideals.

(2) Any s.s. Lie gp. with finite center has a fin. no. of conn. normal sbgps.

Proposition $\mathfrak{g}$ Lie algebra. TFAE:

1) $\mathfrak{g}$ semisimple
2) $\mathfrak{g}$ has no Ab. ideals
3) $\mathfrak{g}$ has no solvable ideals

Pf (1) $\Rightarrow$ (2) defn.

(2) $\Rightarrow$ (3) $\mathfrak{h}$ solvable ideal $\Rightarrow$

$\Rightarrow \mathfrak{h} \supset \mathfrak{h}^{(1)} \supset \dots \supset \mathfrak{h}^{(n-1)} \supset \mathfrak{h}^{(n)} = \{0\}$

$\Rightarrow \mathfrak{h}^{(n-1)}$ is an Abelian ideal in $\mathfrak{h}$.

But $\mathfrak{h}^{(i)}$ are char. ideals $\Rightarrow$

$\Rightarrow \mathfrak{h}^{(n-1)}$ Ab. ideal in $\mathfrak{g}$.

(3) $\Rightarrow$ (1) Want to see that $B_{\mathfrak{g}}$ is non-deg. Let $\mathfrak{h} \subset \mathfrak{g}$ be the kernel of $B_{\mathfrak{g}}$. Since $B_{\mathfrak{g}}$ is $\mathrm{ad}_{\mathfrak{g}}$-invariant $\Rightarrow$ $\mathfrak{h}$ is an ideal and $B_{\mathfrak{h}} = B_{\mathfrak{g}}|_{\mathfrak{h} \times \mathfrak{h}}$. By defn. of $\mathfrak{h}$ $\Rightarrow B_{\mathfrak{h}} \equiv 0 \Rightarrow \mathfrak{h}$ solvable

By hyp. $\Rightarrow \mathfrak{h} = \{0\} \Rightarrow B_{\mathfrak{g}}$ is non-degenerate ∎

Corollary (1) G conn. simple Lie gp. $\Leftrightarrow$ every connected proper normal sbgp. is trivial.
In particular $Z(G)$ is discrete.

(2) G conn. semisimple Lie gp. $\Leftrightarrow$ no connected normal Abelian sbgps $\Leftrightarrow$ no conn. normal solvable subgps.

Proposition $\mathfrak{g}$ semisimple $\Rightarrow$ $[\mathfrak{g},\mathfrak{g}]=\mathfrak{g}$.

Pf $\mathfrak{g} = \sum_{i\in I}^{\oplus} \mathfrak{g}_i$, $|I|<\infty$,
$\mathfrak{g}_i$ simple ideals.

If $i\neq j \Rightarrow [\mathfrak{g}_i,\mathfrak{g}_j] \subset \mathfrak{g}_i\cap\mathfrak{g}_j=\{0\}$

$[\mathfrak{g}_j,\mathfrak{g}_j]$ ideal in $\mathfrak{g}_j$

Since $\mathfrak{g}_i$ not Abelian $\Rightarrow [\mathfrak{g}_i,\mathfrak{g}_i]=\mathfrak{g}_i$

$\Rightarrow [\mathfrak{g},\mathfrak{g}]=\mathfrak{g}$. ∎

LEVI DECOMPOSITION

Ex $GL(n,\mathbb{R})$ not semisimple since $Z(GL(n,\mathbb{R}))=\{\lambda I : \lambda\in\mathbb{R}\}$ & not solvable

Ex $GL(n,\mathbb{R})\ltimes\mathbb{R}^n = \mathrm{Aff}(\mathbb{R}^n)$

$(A,v)\,x = Ax+v$.

Aside: The semidirect product of Lie gps is a Lie gp.

Lemma $\mathfrak{a},\mathfrak{b}\subset\mathfrak{g}$ solvable ideals $\Rightarrow$ $\mathfrak{a}+\mathfrak{b}$ solvable ideal.

Pf $0 \to \underset{\text{solv.}}{\mathfrak{a}} \hookrightarrow \mathfrak{a}+\mathfrak{b} \twoheadrightarrow (\mathfrak{a}+\mathfrak{b})/\mathfrak{a} \to 0$

$(\mathfrak{a}+\mathfrak{b})/\mathfrak{a} \cong \mathfrak{b}/(\mathfrak{a}\cap\mathfrak{b})$ solv. solv.

solv.

Corollary $\forall$ Lie algebra $\mathfrak{g}$ $\exists!$ maximal solvable ideal $\mathfrak{r}\subseteq\mathfrak{g}$ and $\mathfrak{g}/\mathfrak{r}$ is semisimple.
$\mathfrak{r}$ is called the solvable radical.

Pf Only need to show that $\mathfrak{g}/\mathfrak{r}$ is s.s. Let $\mathfrak{h}/\mathfrak{r}\subset\mathfrak{g}/\mathfrak{r}$ a solvable ideal $\Rightarrow \mathfrak{h}\subset\mathfrak{g}$ ideal and since $\mathfrak{r}$, $\mathfrak{h}/\mathfrak{r}$ are solvable $\Rightarrow \mathfrak{h}$ is solvable $\Rightarrow \mathfrak{h}\subset\mathfrak{r} \Rightarrow$
$\Rightarrow \mathfrak{h}/\mathfrak{r}=\{0\}$ ∎

Corollary G connected Lie gp, R conn. sbgp. with $\mathrm{Lie}(R)=\mathfrak{r}=$ solvable rad. of $\mathrm{Lie}(G)=\mathfrak{g}$.
Then R is a conn. solvable closed normal subgp and G/R is s.s.

Pf Only need to show that R is closed.

$R \longleftrightarrow \mathrm{Lie}(R)=\mathfrak{r}$

$\overline{R} \longleftrightarrow \mathrm{Lie}(\overline{R})=\mathfrak{r}'$

Only need to show that $\mathfrak{r}'$ solvable

Rk $H<G$ conn. Lie sbgp.

~~H solvable ⟺~~ $\mathfrak{h}$ solv $\Leftrightarrow \mathrm{ad}(\mathfrak{h}^{\mathbb{C}})$ UT

$\Leftrightarrow \mathrm{ad}(\mathfrak{h})^{\mathbb{C}}$ UT $\Leftrightarrow \mathrm{Ad}(H)^{\mathbb{C}}$ UT

$\mathfrak{r}$ solvable $\Leftrightarrow \mathrm{Ad}(R)^{\mathbb{C}}$ UT

$\Downarrow$

$\mathfrak{r}'$ solvable $\Leftrightarrow \mathrm{Ad}(\overline{R})^{\mathbb{C}}$ UT

Thm (Levi decomposition)
$\mathfrak{g}$ f.d. Lie algebra $\Rightarrow \exists$ a semisimple subalgebra $\mathfrak{s}\subseteq\mathfrak{g}$ s.t.

- $\mathfrak{g}=\mathfrak{s}\oplus\mathfrak{r}$ as vector spaces
- $\mathfrak{g}/\mathfrak{r}\simeq\mathfrak{s}$ as Lie algebras

$\mathfrak{s}$ is called Levi factor, semisimple factor, Levi comp.

The ideal $\mathfrak{r}$ is canonically determined by $\mathfrak{g}$, but if $\mathfrak{s}\subseteq\mathfrak{g}$ is a Levi factor and $\varphi\in\mathrm{Aut}(\mathfrak{g})$ then $\varphi(\mathfrak{s})$ is also a Levi factor.
Moreover $\mathfrak{s}$ (up to auto) is maximal w.r.t. semisimplicity.

Semidirect product of Lie alg.

$\mathfrak{h},\mathfrak{n}$ Lie alg., $\rho:\mathfrak{h}\to\mathrm{Der}(\mathfrak{n})$ homo $\Rightarrow \mathfrak{h}\ltimes_\rho\mathfrak{n}$ is the

vector space $\mathfrak{h} \times \mathfrak{n}$ with bracket
$[(H_1,N_1),(H_2,N_2)] = ([H_1,H_2], [\rho(H_2)N_1, N_2])$.
$\mathfrak{h} \ltimes_\rho \mathfrak{n}$ is a Lie algebra in which $\mathfrak{n}$ is an ideal.

Lemma $\mathfrak{g}$ Lie alg., $\mathfrak{h} \subset \mathfrak{g}$ subalg., $\mathfrak{n} \subset \mathfrak{g}$ ideal. TFAE:

(1) $\exists$ Lie alg. homo $\rho : \mathfrak{h} \to \mathrm{Der}(\mathfrak{n})$ s.t. $\mathfrak{g} = \mathfrak{h} \ltimes_\rho \mathfrak{n}$

(2) $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}$

(3) $\mathfrak{g}$ is a Lie algebra ext. of $\mathfrak{n}$ by $\mathfrak{h}$ that is $\exists$ a short exact sequence
$$0 \to \mathfrak{n} \hookrightarrow \mathfrak{g} \twoheadrightarrow \mathfrak{h} \to 0$$
that splits, that is, if $p : \mathfrak{g} \to \mathfrak{g}/\mathfrak{n}$, $i : \mathfrak{h} \to \mathfrak{g}$ $\Rightarrow$ $p \circ i : \mathfrak{h} \to \mathfrak{g}/\mathfrak{n}$ is a Lie algebra homo.

Fact If $G = H \ltimes_\eta N$, $\eta : H \to \mathrm{Aut}(N)$ $\Rightarrow$ $\mathfrak{g} = \mathfrak{h} \ltimes_\rho \mathfrak{n}$ is a semidirect product with $\rho = d_e\eta$.

Levi's thm $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{r}$ as v.s. but $\mathfrak{g} = \mathfrak{s} \ltimes_\rho \mathfrak{r}$ as Lie alg. with respect to some $\rho : \mathfrak{s} \to \mathrm{Der}(\mathfrak{r})$.

Tomorrow

- 2 Ex. of Levi dec (alg., gp)
- compact Lie gps
- advertisment for symmetric spaces.

26, 27 Jan, 28 Jan morn.