17 December 2020

- ex. of Levi decomp.
- compact Lie gps
- symm. spaces

$\mathfrak g$ Lie algebra $\Rightarrow$ $\exists$ max. solvable ideal $\mathfrak r$ (canonically defined) & s.t. $\mathfrak g/\mathfrak r$ is semisimple (and max w.r.t. semisimplicity) and $\mathfrak g = \mathfrak s \ltimes \mathfrak r$, where $\mathfrak s \cong \mathfrak g/\mathfrak r$ as a Lie alg. and is defn. only up to auto.

Ex $V\subset \mathbb R^n$ subspace,
$\mathfrak g = \{X\in \mathfrak{gl}(n,\mathbb R) : XV\subset V\} = \left\{\begin{pmatrix} * & * \\ 0 & *\end{pmatrix}\right\}$
let $\mathfrak n = \left\{\begin{pmatrix}0 & *\\ 0 & 0\end{pmatrix}\right\}$ $\quad \mathfrak a = \left\{\begin{pmatrix}\lambda_1 I & 0\\ 0 & \lambda_2 I\end{pmatrix}\right\}$.

- $\mathfrak n$ nilpotent ideal
- $\mathfrak a$ Abelian subalgebra

$\Rightarrow [\mathfrak n+\mathfrak a, \mathfrak g]\subset \mathfrak n \subset \mathfrak n+\mathfrak a \Rightarrow$

$\Rightarrow \mathfrak n+\mathfrak a$ is an ideal.

$[\mathfrak n+\mathfrak a, \mathfrak n+\mathfrak a]\subset\mathfrak n \Rightarrow$

$(\mathfrak n+\mathfrak a)^{(1)}$ is nilpotent $\Rightarrow \mathfrak n+\mathfrak a$ solvable.

If $\dim V = k \Rightarrow \mathfrak g/(\mathfrak n+\mathfrak a) \cong \begin{pmatrix}\mathfrak{sl}(k,\mathbb R) & 0\\ 0 & \mathfrak{sl}(n-k,\mathbb R)\end{pmatrix} \cong$

$\cong \mathfrak{sl}(k,\mathbb R)\times\mathfrak{sl}(n-k,\mathbb R)$ semisimple

$\Rightarrow \mathfrak n+\mathfrak a$ is the solvable radical of $\mathfrak g$ and $\mathfrak s = \mathfrak{sl}(k,\mathbb R)\times \mathfrak{sl}(n-k,\mathbb R)$ is a Levi factor.

Corollary G simply conn. Lie gp & R solvable radical $\Rightarrow \exists$ s.s. simply conn. Lie subgp $S < G$ s.t. $G = S \ltimes R$ as Lie gps.

Ex. $G' = GL(n,\mathbb R)\ltimes\mathbb R^n = \mathrm{Aff}(\mathbb R^n)$

$Z(G)\cong\mathbb R \quad \overset{\shortparallel}{G} \Rightarrow GL(n,\mathbb R)$ not s.s.

$\Rightarrow GL(n,\mathbb R)\ltimes \mathbb R^n$ is not the Levi decomp. of $\mathrm{Aff}(\mathbb R^n)$. Let $A=\{\lambda I : \lambda\in\mathbb R\}$ $(= Z(G))$. Then $R = A\ltimes\mathbb R^n$ is the solvable radical and

$GL(n,\mathbb R)\ltimes \mathbb R^n / A\ltimes \mathbb R^n \cong SL(n,\mathbb R)$

which is the Levi factor of $\mathrm{Aff}(\mathbb R^n)$

$\Rightarrow \mathrm{Aff}(\mathbb R^n) = SL(n,\mathbb R)\ltimes(A\ltimes\mathbb R^n)$.

[yesterday: $\mathfrak a, \mathfrak n$ solv. ideals in $\mathfrak g$ $\Rightarrow \mathfrak a+\mathfrak n$ solv. ideal in $\mathfrak g$
$\Rightarrow \exists$ max. solv. ideal $\mathfrak r$ and $\mathfrak g/\mathfrak r$ is semisimple.]

COMPACT GROUPS

G compact Lie group, V fin. dim. v.s. $\pi : G \to GL(V)$. Then π is equivalent to (that is, it can be conjugate into) an orthogonal repr. (i.e. a repr. with values into the orth. gp.). In fact, if $(\ ,\ )$ is any inner product on $V \Rightarrow$

$$\langle v,w\rangle := \int_G (\pi(g)v,\pi(g)w)\,d\mu(g),$$

where μ is the Haar m on G, is a positive defn. inner product that is G-invariant.

Corollary Any compact subgp K of $GL(n,\mathbb R)$ is conj. to a subgp of $O(n,\mathbb R)$.

Pf $\pi = i : K\hookrightarrow GL(n,\mathbb R) \Rightarrow$

$\Rightarrow K < O(\mathbb R^n, \langle\ ,\ \rangle)$ & all orth. rep. are conj. one into another

$\Rightarrow K < O(n,\mathbb R)$. ∎

Corollary $O(n,\mathbb R)$ is a maximal compact subgp. of $GL(n,\mathbb R)$ and is unique up to conjugation.

Proposition G connected s.s. Lie gp.
TFAE:
(1) G is compact
(2) $B_{\mathfrak g}$ is negative definite
(3) $B_{\mathfrak g}$ is definite

Pf (1) $\Rightarrow$ (2) G conn. cpt. s.s $\Rightarrow$ $Ad_G(G)$ is compact and since $Ad_G(G) < GL(\mathfrak g) \Rightarrow Ad_G(G) < O(\mathfrak g,\langle\ ,\ \rangle)$

$\Rightarrow ad_{\mathfrak g}(\mathfrak g)\subset \mathfrak o(\mathfrak g,\langle\ ,\ \rangle) \Rightarrow$

if $A\in ad_{\mathfrak g}(\mathfrak g) \Rightarrow A + A^* = 0$.

let $A := ad_{\mathfrak g}(X)$, $X\in\mathfrak g$

$B_{\mathfrak g}(X,X) = tr(A^2) = \sum_{i,j} A_{ij}A_{ji} =$

$= -\sum_{i,j} A_{ij}^2 \le 0$ with

$B_{\mathfrak g}(X,X)=0 \iff A=\mathrm{ad}_{\mathfrak g}(X)=0 \iff$
$\iff X\in \mathfrak z(\mathfrak g)=\{0\}$ (G s.s.)

(2) $\Rightarrow$ (3)

(3) $\Rightarrow$ (1) $B_{\mathfrak g}$ defnite $\Rightarrow O(\mathfrak g, B_{\mathfrak g})$ is a compact gp. Since $\mathrm{Ad}_G(G)<O(\mathfrak g,B_{\mathfrak g})$ $\Rightarrow \mathrm{Ad}_G(G)$ is compact (and semisimple, since G is s.s.) $\Rightarrow$ G compact.

$G/_{Z(G)} \cong \mathrm{Ad}_G(G)$.

Theorem G compact and semisimple then $\pi_1(G)$ is finite hence $\tilde G$ is compact.

Proposition G cpt. Then $\mathfrak g=\mathfrak z(\mathfrak g)\oplus \mathfrak g'$, where $\mathfrak g'$ is s.s. Moreover $[\mathfrak g,\mathfrak g]=\mathfrak g'$ and if $Z(G)$ is finite $\Rightarrow G$ is semisimple.

Pf $\mathfrak g' := \{X\in\mathfrak g : \langle X,Y\rangle=0 \ \forall Y\in \mathfrak z(\mathfrak g)\}$
$=\mathfrak z(\mathfrak g)^\perp$, where

$\mathrm{Ad}_G(G)<O(\mathfrak g,\langle\,,\rangle)$

Need to show that $B_{\mathfrak g'}$ non-deg.

$\mathfrak z(\mathfrak g)$ ideal, $\langle\,,\rangle$ Ad_G-inv $\Rightarrow$
$\Rightarrow \mathfrak z(\mathfrak g)^\perp=\mathfrak g'$ ideal $\Rightarrow$
$\Rightarrow B_{\mathfrak g'}=B_{\mathfrak g}|_{\mathfrak g'\times\mathfrak g'}$ and this is non-deg. since

$B_{\mathfrak g'}(X,X)=0 \iff \mathrm{ad}_{\mathfrak g'}(X)=0$
$\iff X\in \ker \mathrm{ad}_{\mathfrak g} = \mathfrak z(\mathfrak g)=\{0\}$

Rk $B_{\mathfrak g'}$ is defn., not only non-degenerate.

Corollary G cpt conn. Lie gp. Then $G=TK$, where T, K are closed conn. normal sbgps, $T<Z(G)$, K compact & semisimple and $|T\cap K|<\infty$.

Rk G is an almost direct product

"Pf" $\mathfrak g=\mathfrak z(\mathfrak g)\oplus\mathfrak g' \Rightarrow T$ is s.t. $\mathrm{Lie}(T)=\mathfrak z(\mathfrak g)$, that is $T=Z(G)^0$

hence closed and normal.

To show that K is closed:

G cpt $\therefore$ $\mathfrak g=\mathfrak z(\mathfrak g)\oplus\mathfrak g'$ & $\mathfrak g'$ semisimple, $B_{\mathfrak g'}$ definite $\Rightarrow$
$\Rightarrow$ the conn. sbgp. K s.t. $\mathrm{Lie}(K)=\mathfrak g'$ is compact. ◻

Sketch of the proof G cpt s.s.
$\Rightarrow \pi_1(G)$ finite.

- M cpt mfd $\Rightarrow \pi_1(M)$ f.g.
- G top. gp $\Rightarrow \pi_1(G)$ Abelian

$\Big\}\Rightarrow$

$\Rightarrow G$ cpt. Lie gp $\Rightarrow \pi_1(G)$ f.g. Abel.

$$\pi_1(G)\cong \sum_{i=1}^{\ell}{}^{\oplus}\mathbb Z \oplus \sum_{j=1}^{q} \mathbb Z/n_j\mathbb Z$$

Want to show: $\ell=0$.

$H_1(M,\mathbb Z)$ = singular hom. of a mfd M with coeff $\mathbb Z$.

$$\Rightarrow \pi_1(M)\big/[\pi_1(M),\pi_1(M)] \cong H_1(M,\mathbb Z).$$

In our case

$\pi_1(G)\cong H_1(M,\mathbb Z)$

Universal Coefficient thm.

$H^1(G,\mathbb R)\cong \mathrm{Hom}(H_1(G,\mathbb Z),\mathbb R)=$

$=\mathrm{Hom}(\pi_1(G),\mathbb R)$

$=\mathrm{Hom}\left(\sum_{i=1}^{\ell}{}^{\oplus}\mathbb Z\oplus \sum_{j=1}^{q}\mathbb Z/n_j\mathbb Z, \mathbb R\right)$

$=\mathrm{Hom}\left(\sum^{\ell}{}^{\oplus}\mathbb Z,\mathbb R\right)\cong \mathbb R^\ell$

Enough to show $H^1(G,\mathbb R)=0$

$$H^*(G,\mathbb{R}) \simeq H^*_{dR}(G) \simeq H\left(\Omega^*(G)^G\right)$$

G-invariant diff. forms on G.

where if $\omega \in \Omega^1(G)$, $d\omega \in \Omega^2(G)$ is defn. as

$$d\omega(X,Y) := X(\omega(Y)) + Y(\omega(X)) - \omega([X,Y])$$

$X, Y \in Vect(G)$

To show that $H^1(G,\mathbb{R}) = 0$ it is enough to show that if $d\omega = 0$ then $\omega = 0$. Let $X_e, Y_e \in \mathfrak{g}$, $X, Y \in Vect(G)^G$. By inv. of ω, X, Y, the functions $\omega(X)$, $\omega(Y)$ are constant $\Rightarrow$

$$0 = d\omega(X,Y)_e = X(\omega(Y))_e + Y(\omega(X))_e - \omega([X,Y])_e = -\omega([X,Y])_e$$

$\mathfrak{g}$ s.s. $\Rightarrow [\mathfrak{g},\mathfrak{g}] = \mathfrak{g} \Rightarrow \omega \equiv 0$! ☒

SYMMETRIC SPACES

Examples

(1) Euclidean space $(\mathbb{R}^n, g_{eucl}) = \mathbb{E}^n$

$Iso(\mathbb{E}^n) = O(n) \ltimes \mathbb{R}^n$

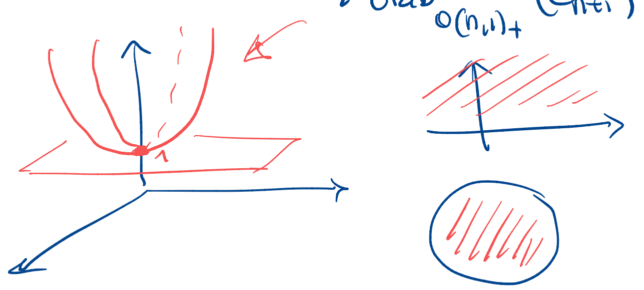
(2) (S^n, g_{eucl})

(3) $\mathcal{H}^n = \{x \in \mathbb{R}^{n+1} : q(x,x) = -1,\ x_{n+1} > 0\}$

$$q(x,y) = \sum_{i=1}^{n} x_i y_i - x_{n+1} y_{n+1}$$

$Iso(\mathcal{H}^n) = O(n,1)_+ \curvearrowright \mathcal{H}^n$ transitively

$\Rightarrow \mathcal{H}^n = O(n,1)_+ / Stab_{O(n,1)_+}(e_{n+1})$



(4) $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$ (cmp with $\mathbb{R}^n$)

(5) Σ_g cpt conn. oriented surface of genus $g \geq 2$ $\Rightarrow$

$\Rightarrow \Sigma_g = \mathcal{H}^2/\Gamma$, $\Gamma < Iso(\mathcal{H}^2)$ lattice

(6) Quotient of $\mathcal{H}^2$ by $SL(2,\mathbb{Z})$ or by any f.i. subgp. of $SL(2,\mathbb{Z})$ $\rightsquigarrow$ quotient is a f.v. surface not cocompact

(7) Any Riem. symm. space with s.s. isom. gp admits a quotient which is cpt and one that finite volume & non-cpt.

Rk (1), (2), (3) are globally s.s.

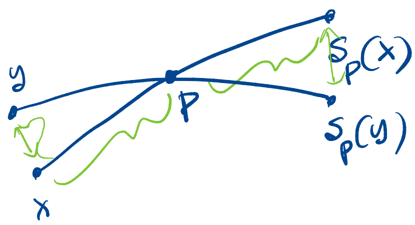
(4) ÷ (7) locally s.s.

that is quotient of glob. ss. by an appr. discrete sbgp.

Riem. geom. ch. of s.s.

M Riem. mfd.

A geodesic symmetry at $p \in M$ is a map s_p defn. in a nbd of p. that reverses local geod. through p.



Defn A Riem. mfd is a Riem. loc. symm. space if $\forall p \in M$ $\exists s_p$ that is an isometry.

The symm. space is global if the geod. symm. are defn. everywhere.

loc. symm. sp. $\longleftrightarrow$ number theory

symm. sp $\longleftrightarrow$ Lie theory

Why Lie gps?

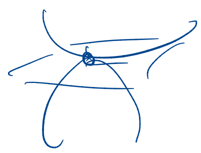
Iso(sym. space) is a f.d. Lie gp, small enough to be f.d. but large enough to be acting transit. on the sym. sp.

$\Rightarrow X = G/K, \quad G = \mathrm{Iso}(X)$ Lie gp.

Alg. defn.

G conn. Lie gp, $\sigma: G \to G$ invol. autom. ($\sigma^2 = \mathrm{Id}$). A Riem. s.s. is a hom. space G/K, where

$$(G^\sigma)^\circ < K < G^\sigma, \quad K \text{ cpt.}$$



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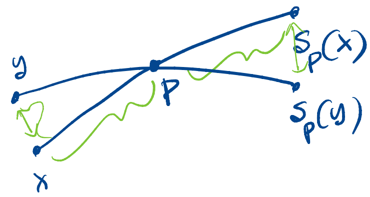

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