ETH
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich
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Functional Analysis I
Final Exam

Name:
Student number:
Study program:

- Put your student identity card onto the desk.
- During the exam no written aids nor calculators or any other electronic device are allowed in the exam room. **Phones must be switched off and stowed away** in your bag during the whole duration of the exam.
- A4-paper is provided. No other paper is allowed. Write with blue or black pens. **Do *not* use pencils, erasable pens, red or green ink, nor Tipp-Ex.**
- Start every problem on a new sheet of paper and write your name on every sheet of paper. Leave enough ($\approx 3\,\text{cm}$) empty space on the margins (top, bottom and sides). You can solve the problems in any order you want, but please sort them in the end.
- You will be asked to return **all** sheets of paper you are assigned, however you have the freedom to clearly cross those sheets you do not want us to consider during the grading process (i. e. scratch paper).
- Please write neatly! Please do not put the graders in the unpleasant situation of being incapable of reading your solutions, as this will certainly not play in your favour!
- All your answers do need to be properly justified. It is fine and allowed to use theorems/statements proved in class or in the homework (i. e. in problem sets 1–13) without reproving them (**unless otherwise stated**), but you should provide a precise statement of the result in question.
- Said $x_i \in \{0, \dots, 10\}$ your score on exercise i, your grade will be bounded from below by $\min\{6, \frac{1}{10}\sum_{i=1}^{7} x_i\}$.
The *complete and correct* solution of 6 problems out of 7 is enough to obtain the maximum grade 6,0.
The *complete and correct* solution of 4 problems out of 7 is enough to obtain the pass/sufficient grade 4,0.
- The duration of the exam is **180 minutes**.

Do not fill out this table!

task	points	check
1	[10]	
2	[10]	
3	[10]	
4	[10]	
5	[10]	
6	[10]	
7	[10]	
total	[70]	

grade:

Problem 1. **[10 points]**

Let $(X, \|\cdot\|_X)$ be a Banach space over the real field $\mathbb{R}$.

(a) Define the weak topology τ_w on X.

(b) Prove that $\tau_w \subset \tau$ where τ is the topology induced by the Banach norm $\|\cdot\|_X$.

(c) Prove that a (linear) subspace of a Banach space is closed in the norm topology if and only if it is closed in the weak topology. (If you wish you can appeal to a suitable separation theorem without reproving it, but in that case you do need to state it precisely.)

Problem 2. **[10 points]**

Let $T\colon \ell^2 \to \ell^2$ be the *double-shift* operator defined by $(Tx)_n = x_{n+2}$ for $x \in \ell^2$.

(a) Determine the operator norm of T.

(b) Determine the adjoint T^* of T.

(c) Determine the eigenvalues of T.

Problem 3. **[10 points]**

Let $(X, \|\cdot\|_X)$ be a reflexive Banach space over $\mathbb{R}$. Let $\ell \in X^*$ and for any given real number $\alpha \geq 0$ consider the functional $F_\alpha\colon X \to \mathbb{R}$ given by

$$F_\alpha(x) = \|x\|_X^2 - |\ell(x)|^\alpha.$$

Prove that there exists $\alpha_0 > 0$ (to be explicitly determined) such that:

- For any $0 \leq \alpha < \alpha_0$ the functional F_α has a global minimum on X, namely the value $\inf_{x\in X} F_\alpha(x)$ is a real number attained by F_α at some (not necessarily unique) $\overline{x} \in X$.
- For any $\alpha > \alpha_0$ there exist examples of reflexive Banach spaces $(X, \|\cdot\|_X)$ and linear functionals $\ell \in X^*$ such that one has instead $\inf_{x\in X} F_\alpha(x) = -\infty$.

Problem 4. **[10 points]**

(a) Determine all values of $p \in [1, \infty]$ such that the Banach space $L^p(\mathbb{R})$ is actually Hilbertean (meaning that the norm $\|\cdot\|_{L^p}$ is induced by a scalar product).

(b) Determine all values of $p \in [1, \infty]$ such that the Banach space $\ell^p(\mathbb{N})$ is actually Hilbertean (meaning that the norm $\|\cdot\|_{\ell^p}$ is induced by a scalar product).

It is advised not to forget the case $p = \infty$ in your discussion.

Problem 5. **[10 points]**

Let $(H, \langle\cdot,\cdot\rangle)$ be an infinite-dimensional Hilbert space over $\mathbb{C}$ and let $A\colon H \to H$ be linear.

(a) Give a complete proof that, if A is self-adjoint then it is continuous i. e. $A \in L(H)$. (You may appeal to the closed graph theorem without reproving it, but in that case you do need to state it precisely.)

(b) Define the spectrum $\sigma(A)$ of an element $A \in L(H)$.

(c) Let A be compact, self-adjoint and injective. Prove that $0 \in \sigma(A)$ and determine whether 0 belongs to the point spectrum $\sigma_p(A)$, to the continuous spectrum $\sigma_c(A)$ or to the residual spectrum $\sigma_r(A)$ of A.

Problem 6. **[10 points]**

(a) State the two equivalent forms of Baire's Lemma. Then present an explicit example which shows that the conclusion fails if the metric space question is not assumed to be complete.

(b) Let $\Omega \subset (0,\infty)$ be an unbounded open set. Consider, further, the set

$$\Sigma := \{x \in (0,\infty) \ : \ nx \in \Omega \text{ for infinitely many } n \in \mathbb{N}\}.$$

Prove that Σ is dense in $(0,\infty)$.

Problem 7. **[10 points]**

Suppose that X, Y, Z are Banach spaces over $\mathbb{R}$, let $P \in L(X;Y)$ and assume that the inclusion $X \subset Z$ is compact.

Suppose also that there is a constant $C > 0$ such that for all $x \in X$ one has

$$\|x\|_X \le C\Big(\|Px\|_Y + \|x\|_Z\Big) \tag{$*$}$$

(a) If P is injective, show that there is another constant $C' > 0$ such that for all $x \in X$ one has

$$\|x\|_X \le C'\|Px\|_Y.$$

(b) Without assuming that P is injective show that $(*)$ implies that $\ker(P)$ has finite dimension. Hence, prove the existence of a closed subspace W of X with $X = \ker(P) \oplus W$ (i. e. a topologically complementary subspace W of $\ker(P)$ in X). Then exploit part (a) to show that for all $x \in W$ one has

$$\|x\|_X \le C''\|Px\|_Y$$

for some constant $C'' > 0$.