Problem 1.

(a) $\Omega \in \tau_w \;\Leftrightarrow\; \Omega$ is an arbitrary union of finite intersections of sets of the form

$$\Omega_{f,U} = f^{-1}(U), \quad f \in X^*, \quad U \subset \mathbb{R} \text{ open}.$$

(b) Since $f \in X^*$ is continuous with respect to τ, we have $\Omega_{f,U} \in \tau$ for all $f \in X^*$ and $U \subset \mathbb{R}$ open. This implies $\tau_w \subset \tau$.

(c) *Separation Theorem.* Let $M \subset X$ be a closed subspace. Given $x_0 \in X \setminus M$, with

$$d = \inf_{x \in M} \|x_0 - x\|_X > 0,$$

there exists $f \in X^*$ with $f|_M = 0$, $\|f\|_{X^*} = 1$ and $f(x_0) = d$.

Let $M \subset X$ be a linear subspace.

First Solution. By part (b), if M is closed in the weak topology, it is closed in the norm topology.

Conversely, let us assume that M is closed in the norm topology: we will now prove that $X \setminus M$ is open in the weak topology. Indeed, let $x_0 \in X \setminus M$: by the separation Theorem above, there is a linear form $f \in X^*$ separating $\{x_0\}$ from M. The preimage $f^{-1}((\frac{d}{2}, \infty))$ is an open neighbourhood of x_0 in the weak topology that is disjoint from M.

Second Solution. Let $M \subset X$. Part (b) implies $M \subset \overline{M} \subset \overline{M}_w$. Thus, $M = \overline{M}_w \Rightarrow M = \overline{M}$.

Let $M \subset X$ be a closed subspace. By contradiction, suppose there exists $x_0 \in \overline{M}_w \setminus M$. Let $f \in X^*$ be as given by the Separation Theorem. Then, $f^{-1}(\{0\})$ is closed in the weak topology and contains M but not x_0. This contradicts $x_0 \in \overline{M}_w$.

Problem 2.

$$\begin{aligned} T\colon \ell^2 &\to \ell^2 \\ (x_0, x_1, x_2, \dots) &\mapsto (x_2, x_3, x_4, \dots) \end{aligned}$$

(a) For every $x \in \ell^2$, there holds

$$\|Tx\|_{\ell^2} \leq \|x\|_{\ell^2} \qquad \Rightarrow \|T\| \leq 1.$$

For $e_3 = (0, 0, 1, 0, \dots) \in \ell^2$, there holds $\|e_3\|_{\ell^2} = 1 = \|Te_3\|_{\ell^2}$. Therefore, $\|T\| = 1$.

(b) Since for any $x, y \in \ell^2$

$$\langle Tx, y\rangle_{\ell^2} = \sum_{n=0}^{\infty} x_{n+2} y_n = \sum_{n=2}^{\infty} x_n y_{n-2}$$

we conclude

$$\begin{aligned} T^*\colon \ell^2 &\to \ell^2 \\ (y_0, y_1, \dots) &\mapsto (0, 0, y_0, y_1, \dots). \end{aligned}$$

(c) Suppose, $\lambda \in \mathbb{R}$ is an eigenvalue of T with eigenvector $x \in \ell^2$. Let x_m be the first non-zero entry of x. Then, $Tx = \lambda x$ implies $x_{m+2} = \lambda x_m$.

Inductively, we obtain $x_{m+2k} = \lambda^k x_m$ for every $k \in \mathbb{N}$.

$$\Rightarrow \ \|x\|_{\ell^2}^2 \geq \sum_{k\in\mathbb{N}} |x_{m+2k}|^2 = |x_m|^2 \sum_{k\in\mathbb{N}} |\lambda|^{2k} \qquad \Rightarrow |\lambda| < 1.$$

Let $\lambda \in \mathbb{R}$ with $|\lambda| < 1$. Consider $x = (x_n)_{n\in\mathbb{N}}$ given by

$$x_n = \begin{cases} \lambda^{\frac{n}{2}} & \text{for even } n, \\ 0 & \text{for odd } n. \end{cases}$$

Then, $x \in \ell^2$ because $|\lambda| < 1$ implies that

$$\|x\|_{\ell^2}^2 = \sum_{n\in\mathbb{N} \text{ even}} |\lambda^{\frac{n}{2}}|^2 \leq \sum_{n\in\mathbb{N}} |\lambda|^n < \infty.$$

Moreover,

$$Tx = T(0, \lambda, 0, \lambda^2, \dots) = (0, \lambda^2, 0, \lambda^3, \dots) = \lambda(0, \lambda, 0, \lambda^2, \dots) = \lambda x.$$

Hence, every $\lambda \in \mathbb{R}$ with $|\lambda| < 1$ is an eigenvalue of T.

Problem 3.

We want to show that $\alpha_0 = 2$.

For every $0 \le \alpha < 2$, the map F_α is coercive, because

$$|F_\alpha(x)| = \|x\|_X^2 - |\ell(x)|^\alpha \ge \Big(\|x\|_X^{2-\alpha} - \|\ell\|_{X^*}^\alpha\Big)\|x\|_X^\alpha \xrightarrow{\|x\|_X \to \infty} \infty$$

As proven in class, $x \mapsto \|x\|_X^2$ is weakly sequentially lower semi-continuous. Moreover, by definition of weak convergence $x \mapsto \ell(x)$ is weakly sequentially continuous. Therefore, F_α is weakly sequentially lower semi-continuous.

Since X is reflexive, the Direct Method ["Variationsprinzip", Satz 5.4.1] applies and we obtain that

$$\exists x_0 \in X \quad F_\alpha(x_0) = \inf_{x \in X} F_\alpha(x).$$

Given $\alpha > 2$, consider the example $(X, \|\cdot\|_X) = (\mathbb{R}, |\cdot|)$ with $\ell(x) = x$.

$$F_\alpha(x) = x^2 - |x|^\alpha = \Big(1 - |x|^{\alpha-2}\Big)|x|^2 \xrightarrow{x \to \infty} -\infty.$$

Recall that $(\mathbb{R}, |\cdot|)$ is reflexive, because any finite dimensional Banach space is reflexive.

Problem 4.
The parallelogram identity reads

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

and it is a necessary and sufficient condition for a Banach space to be Hilbertian.

(a) For any $1 \leq p \leq \infty$, we can consider the characteristic functions $\chi_{[0,1]} \in L^p(\mathbb{R})$ and $\chi_{[1,2]} \in L^p(\mathbb{R})$. Then,

$$\begin{aligned} 2\|\chi_{[0,1]}\|^2_{L^p(\mathbb{R})} &= 2\|\chi_{[1,2]}\|^2_{L^p(\mathbb{R})} = 2 \\ \|\chi_{[0,1]} + \chi_{[1,2]}\|^2_{L^p(\mathbb{R})} &= \|\chi_{[0,1]} - \chi_{[1,2]}\|^2_{L^p(\mathbb{R})} = \begin{cases} 1 & \text{if } p = \infty, \\ 2^{\frac{2}{p}} & \text{else.} \end{cases} \end{aligned}$$

Hence, the parallelogram identity is violated for $p = \infty$. For $1 \leq p < \infty$, the parallelogram identity implies $2^{\frac{2}{p}} + 2^{\frac{2}{p}} = 2 + 2$, hence $p = 2$.

The space $L^2(\mathbb{R})$ is indeed Hilbertean by virtue of the scalar product

$$\langle f, g \rangle_{L^2(\mathbb{R})} = \int_{\mathbb{R}} f(x)g(x)\,dx.$$

(b) For any $1 \leq p \leq \infty$, we can consider the elements $x = (1, 0, 0, \ldots) \in \ell^p$ and $y = (0, 1, 0, \ldots) \in \ell^p$. Then,

$$\begin{aligned} 2\|x\|^2_{\ell^p} &= 2\|y\|^2_{\ell^p} = 2 \\ \|x+y\|^2_{\ell^p} &= \|x-y\|^2_{\ell^p} = \begin{cases} 1 & \text{if } p = \infty, \\ 2^{\frac{2}{p}} & \text{else.} \end{cases} \end{aligned}$$

Hence, the parallelogram identity is violated for $p = \infty$. For $1 \leq p < \infty$, the parallelogram identity implies $2^{\frac{2}{p}} + 2^{\frac{2}{p}} = 2 + 2$, hence $p = 2$.

The space $\ell^2(\mathbb{R})$ is indeed Hilbertean by virtue of the scalar product

$$\langle x, y \rangle_{\ell^2} = \sum_{n \in \mathbb{N}} x_n y_n.$$

Problem 5.

(a) Let H be a Hilbert space over $\mathbb{C}$ and let $A\colon H \to H$ be linear. According to the closed graph theorem,

$$A \in L(H;H) \qquad \Leftrightarrow \qquad A \text{ has closed graph.}$$

Assume that A is self-adjoint. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in H and $x, y \in H$ such that $\|x_n - x\|_H \to 0$ and $\|Ax_n - y\|_H \to 0$ as $n \to \infty$.

For any $z \in H$ we obtain (by continuity of the scalar product and by symmetry of A)

$$\langle y, z\rangle = \lim_{n\to\infty}\langle Ax_n, z\rangle = \lim_{n\to\infty}\langle x_n, Az\rangle = \langle x, Az\rangle = \langle Ax, z\rangle.$$

Hence, $\langle y - Ax, z\rangle = 0$. Choosing $z = y - Ax$, we conclude $Ax = y$.

(b) The spectrum of $A \in L(H;H)$ is $\sigma(A) = \mathbb{C} \setminus \rho(A)$, where

$$\rho(A) := \{\lambda \in \mathbb{C} \mid (\lambda - A)\colon H \to H \text{ bijective}, \exists(\lambda - A)^{-1} \in L(H,H)\}.$$

(c) Let $A\colon H \to H$ be compact, injective and self-adjoint. By the spectral theorem, A has eigenvalues $\lambda_k \in \mathbb{R} \setminus \{0\}$ with $\lambda_k \to 0$ as $k \to \infty$.

Let φ_k with $\|\varphi_k\| = 1$ be the corresponding eigenvectors. The inverse $A^{-1}\colon A(H) \to H$ cannot be continuous because

$$A^{-1}\varphi_k = \frac{1}{\lambda_k}\varphi_k \qquad \Rightarrow \quad \|A^{-1}\varphi_k\|_H \geq \frac{1}{\lambda_k} \xrightarrow{k\to\infty} \infty.$$

Therefore, $0 \in \sigma(A)$.

Alternative. If $(x_n)_{n\in\mathbb{N}} \subset B_1$ is any sequence, then $(Ax_n)_{n\in\mathbb{N}}$ has a convergent subsequence by compactness of A. If A^{-1} is continuous then $(x_n)_{n\in\mathbb{N}} = (A^{-1}Ax_n)_{n\in\mathbb{N}}$ has a convergent subsequence which would imply that $B_1 \subset H$ is compact.

Since $\{\varphi_k \mid k \in \mathbb{N}\}$ is a Hilbertean basis, its span (finite linear combinations) is dense and contained in the image of A. Hence, $0 \in \sigma_c(A)$.

Problem 6.

(a) Let (M,d) be a *complete* metric space. Then *possible statements of Baire are*

- Countable intersections of dense open sets are dense.
- If $\left(\bigcup_{j=1}^{\infty} A_j\right)^\circ \neq \emptyset$ with $A_j \subset M$ closed then there is $j_0 \in \mathbb{N}$ with $(A_{j_0})^\circ \neq \emptyset$.

The statement of Baire fails in the (incomplete) metric space $(\mathbb{Q}, |\cdot|)$: For any $q \in \mathbb{Q}$, the set $\{q\}$ is closed with empty interior but $\mathbb{Q} = \bigcup_{q\in\mathbb{Q}}\{q\}$ is a countable union.

(b) By definition (lim sup of sets), we have

$$\Omega_k := \{x \in (0,\infty) \mid kx \in \Omega\}, \qquad \Sigma = \bigcap_{N=1}^{\infty} \bigcup_{k=N}^{\infty} \Omega_k.$$

Let,

$$A_N := \bigcup_{k=N}^{\infty} \Omega_k = \{x \in (0,\infty) \mid \exists k \geq N: \ kx \in \Omega\}.$$

As union of open sets, A_N is open.

By contradiction, we prove that A_N is also dense: If A_N is not dense in $(0,\infty)$, there exist $a < b$ such that $]a,b[\cap A_N = \emptyset$, which implies

$$\forall k \geq N: \quad]ka, kb[\cap \Omega = \emptyset.$$

For k sufficiently large, $ka < (k-1)b$ which contradicts unboundedness of Ω. By Baire, Σ is dense as a countable intersection of the dense open sets A_N.

Problem 7.

(a) For the sake of a contradiction, assume the claimed inequality is false: thus for any $k \geq 1$ one can find $x_k \in X$ with $\|x_k\|_X = 1$ and $\|Px_k\|_Y \leq \frac{1}{k}$. By compactness of the inclusion $X \subset Z$ one can find $\Lambda \subset \mathbb{N}$ such that

$$x_k \to x_\infty^Z \ \text{ in } \ (Z, \|\cdot\|_Z) \ \ (k \to \infty, \ k \in \Lambda).$$

At this stage, using $(*)$ with $x_l - x_m$ in lieu of x, namely

$$\|x_l - x_m\|_X \leq C\Big(\|P(x_l - x_m)\|_Y + \|x_l - x_m\|_Z\Big).$$

one gets that the sequence $(x_k)_{k\in\Lambda}$ is Cauchy in $(X, \|\cdot\|_X)$ so by completeness $x_k \to x_\infty^X$ in $(X, \|\cdot\|_X)$ $(k \to \infty, \ k \in \Lambda)$. Since $X \subset Z$ continuously (as implied by the definition of compact inclusion, Def. 6.2.1.) one also gets $x_k \to x_\infty^X$ in $(Z, \|\cdot\|_Z)$ $(k \to \infty, \ k \in \Lambda)$ and thus, comparing the two, $x_\infty^X = x_\infty^Z$ by uniqueness of the limit in $(Z, \|\cdot\|_Z)$. Let us then simply denote by x_∞ such limit. Since $P \in L(X; Y)$ we have

$$x_k \to x_\infty \ \Rightarrow \ Px_k \to Px_\infty \ \ (k \to \infty, \ k \in \Lambda)$$

but on the other hand $Px_k \to 0$ by construction, so we conclude $Px_\infty = 0$ and hence, by injectivity $x_\infty = 0$. However it should be $\|x_\infty\|_X = 1$ by the fact that $\|x_k\|_X = 1$ for any $k \geq 1$, which yields a contradiction.

(b) Let us prove that $M := \ker(P)$ has finite dimension by showing that $B_1(0; M)$ is relatively compact in $(X, \|\cdot\|_X)$. To this purpose, pick $(x_k)_{k\in\mathbb{N}}$ a sequence with $\|x_k\|_X < 1$ and let us prove it has a converging subsequence. Observe that inequality $(*)$, when restricted to $x \in M$ takes the form

$$\|x\|_X \leq C\|x\|_Z.$$

Hence (arguing as above) one first gets $x_k \to x_\infty^Z$ $(k \to \infty, k \in \Lambda)$ and then, by the inequality above $(x_k)_{k\in\Lambda}$ is Cauchy in $(X, \|\cdot\|_X)$ hence convergent to x_∞^X (incidentally: $x_\infty^X = x_\infty^Z$ by uniqueness of the limit in Z).

At this stage, the fact that M is topologically complemented in X follows by Problem 7.2, so let us write $X = M \oplus W$ with $W \subset X$ closed (i. e. $\overline{W} = W$) by Problem 3.4.

Lastly, the restricted operator $P^\rho \colon W \to Y$ is linear, bounded and one can invoke the result of part (a). With W in lieu of X and P^ρ in lieu of P to conclude that $\|x\|_X \leq C''\|Px\|_Z$ uniformly for $x \in W \subset X$, which completed the proof.