ETH
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich
Prof. Dr. Alessandro Carlotto 18 December 2020

Functional Analysis I
Probeprüfung

Name:
Student number:
Study program:

- Put your student identity card onto the desk.
- During the exam no written aids nor calculators or any other electronic device are allowed in room. **Phones must be switched off and stowed away** in your bag during the whole duration of the exam.
- A4-paper is provided. No other paper is allowed. Write with blue or black pens. **Do *not* use pencils, erasable pens, red or green ink, nor Tipp-Ex.**
- Start every problem on a new sheet of paper and write your name on every sheet of paper. Leave enough ($\approx$ 2 cm) empty space on the margins (top, bottom and sides). You can solve the problems in any order you want, but please sort them in the end.
- Please write neatly! Do not put the graders in the unpleasant situation of being incapable of reading your solutions, as this will certainly not play in your favour!
- All your answers do need to be properly justified. It is fine and allowed to use theorems/statements proved in class without reproving them (**unless otherwise stated**) but you may be required to provide a precise statement of the result in question.
- Said $x_i \in \{0, \dots, 10\}$ your score on task i, your grade will be bounded from below by $\min\{6, \frac{1}{10}\sum_{i=1}^{7} x_i\}$.
The *complete and correct* solution of 6 problems out of 7 is enough to obtain the maximum grade 6,0.
The *complete and correct* solution of 4 problems out of 7 is enough to obtain the pass/sufficient grade 4,0.
- The duration of the exam is **180 minutes**.

Do not fill out this table!

task	points	check
1	[10]	
2	[10]	
3	[10]	
4	[10]	
5	[10]	
6	[10]	
7	[10]	
total	[70]	

grade:

Problem 1. **[10 points]**

Let $(X, \|\cdot\|_X)$ be a normed space over the real field $\mathbb{R}$. Let $A, B \subset X$ be non-empty, convex and disjoint.

(a) Suppose that A is open. What can you say about the separation of A from B? You are only required to write a precise statement.

(b) Suppose that A is compact and B is closed. Prove that there exists $r > 0$ such that $U_r(A) \cap B = \emptyset$ where by definition $U_r(A) = \bigcup_{a \in A} B_r(a)$.

Given $l \in X^*$, $l \neq 0$ prove that

$$\sup_{a \in A} l(a) < \sup_{a' \in U_r(A)} l(a').$$

(c) Suppose that A is compact and B is closed. Use the two steps above to prove that A and B can be strictly separated (you also need to state the theorem precisely).

Problem 2. **[10 points]**

(a) Let (X, d) be a metric space. Define what it means for a subset $A \subset X$ to be a *first category set.*

(b) Let $\mathcal{H} \subset \ell^2$ be the subspace consisting of those sequences $(x_n)_{n \in \mathbb{N}} \in \ell^2$ satisfying $\sum_{n=0}^{\infty} n^2 |x_n|^2 < \infty$. Show that $\mathcal{H}$ is a first category set in ℓ^2.

Problem 3. **[10 points]**

Let H be a Hilbert space over $\mathbb{R}$ and let $\|\cdot\|$ denote the corresponding induced norm. Consider a continuous, convex map $F \colon H \to \mathbb{R}$ such that

$$\lim_{\|x\| \to +\infty} \frac{|F(x)|}{\|x\|} = +\infty,$$

and consider finitely many elements $\ell_1, \ldots, \ell_N \in H^*$.

(a) Prove that $F \colon H \to \mathbb{R}$ is sequentially lower-semicontinous with respect to the weak topology on H.

(b) Defined

$$G(x) := F(x) - \sum_{i=1}^{N} |\ell_i(x)|,$$

prove that G has a global minimum on H, namely the value $\inf_{x \in H} G(x)$ is a real number attained by G at some (not necessarily unique) $\overline{x} \in H$.

Problem 4. **[10 points]**

Let $g \in L^\infty(\mathbb{R};\mathbb{C})$. Consider the operator $T\colon L^2(\mathbb{R};\mathbb{C}) \to L^2(\mathbb{R};\mathbb{C})$ given by $Tf = fg$.

(a) Compute the operator norm of T.

(b) Show that the spectrum of T is equal to the essential image of g, namely

$$\sigma(T) = \Big\{\lambda \in \mathbb{C} \ : \ |g^{-1}(B_\varepsilon(\lambda))| > 0 \ \forall \varepsilon > 0\Big\}.$$

Problem 5. **[10 points]**

Let H be a Hilbert space over $\mathbb{R}$ and let $L(H)$ denote the space of linear, continuous operators $H \to H$, endowed with the standard operator norm $\|\cdot\|_{L(H)}$. Is it true or false that such norm is always Hilbertean (meaning that the norm $\|\cdot\|_{L(H)}$ is induced by a scalar product)? If true, prove your assertion, else provide a counterexample.

Problem 6. **[10 points]**

(a) Let $(X, \|\cdot\|_X)$ be a normed space. Define what it means that $(X, \|\cdot\|_X)$ is separable. Provide an example of a Banach space that is not separable (you just need to state it, no proof is needed).

(b) Let $(Y, \|\cdot\|_Y)$ be a Banach space. Define what it means that $(Y, \|\cdot\|_Y)$ is reflexive. Provide an example of a Banach space that is not reflexive (you just need to state it, no proof is needed).

(c) Let X be a separable normed space and Y be a reflexive Banach space. Given $(F_n)_{n\in\mathbb{N}}$ a bounded sequence in $L(X;Y)$ prove that there exists a subsequence $(F_{n_k})_{k\in\mathbb{N}}$ such that for all $x \in X$ the sequence $(F_{n_k}x)_{k\in\mathbb{N}}$ weakly converges in Y.

Problem 7. **[10 points]**

Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space and consider an orthogonal decomposition into finite dimensional subspaces of positive dimension, namely $H = \bigoplus_{j=1}^\infty H_j$ (thus each $v \in H$ can be uniquely written as $v = \sum_{j=1}^\infty v_j$ with $v_j \in H_j$ for all $j \geq 1$).

Let $c = (c_1, c_2, \ldots)$ where $c_j > 0$ for all $j \geq 1$, and define the subset

$$A_c := \{v \in H : \ \|v_j\| \leq c_j \ \forall j \geq 1\}.$$

Prove that $c \in \ell^2$ if and only if A_c is compact in H.