Problem 1.

(a) Let (X_1, τ_1) and (X_2, τ_2) be topological spaces. By definition,

$$f\colon X_1 \to X_2 \text{ open} \quad \Leftrightarrow \quad \forall\, \Omega \in \tau_1 : \quad f(\Omega) \in \tau_2.$$

The embedding $\mathbb{R} \hookrightarrow \mathbb{R}^2$ is continuous, injective but not open: Indeed, the set $\mathbb{R} \times \{0\} \subset \mathbb{R}^2$ is not open in $\mathbb{R}^2$.

(b) Statement of the open mapping theorem:

Let X, Y be Banach spaces and let $A\colon X \to Y$ be linear, continuous and surjective. Then A is open.

(c) Towards a contradiction, suppose $T\colon \ell^1 \to \ell^\infty$ is linear, continuous and bijective. By the open mapping theorem, T is open, hence $T^{-1}\colon \ell^\infty \to \ell^1$ is continuous. Consequently, $T\colon \ell^1 \to \ell^\infty$ would be a homeomorphism. This contradicts the fact that ℓ^1 is separable and ℓ^∞ is not.

Problem 2.

(a) The map F is coercive, because $F(x) \geq \|x - x_1\|_X^2 \to \infty$ as $\|x\|_X \to \infty$. Moreover F is weakly sequentially lower semicontinuous because the map $x \mapsto \|x\|_X$ is.

Since X is reflexive, the Direct Method ["Variationsprinzip", Satz 5.4.1] applies and we obtain $\overline{x} \in X$ satisfying

$$F(\overline{x}) = \inf_{x \in X} F(x).$$

(b) Suppose, $\overline{y} \in X \setminus \{\overline{x}\}$ is another minimiser of F and consider $\overline{z} = \frac{1}{2}(\overline{x} + \overline{y})$.

In the Hilbertean case, the parallelogram identity holds and implies

$$\begin{aligned}
\|\overline{z} - x_i\|_X^2 &= \left\| \frac{\overline{x} - x_i}{2} + \frac{\overline{y} - x_i}{2} \right\|_X^2 \\
&= 2\left\| \frac{\overline{x} - x_i}{2} \right\|_X^2 + 2\left\| \frac{\overline{y} - x_i}{2} \right\|_X^2 - \underbrace{\left\| \frac{\overline{x} - x_i}{2} - \frac{\overline{y} - x_i}{2} \right\|_X^2}_{\neq 0} \\
&< \frac{\|\overline{x} - x_i\|_X^2}{2} + \frac{\|\overline{y} - x_i\|_X^2}{2}.
\end{aligned}$$

Hence, a contradiction follows from

$$F(\overline{z}) < \frac{F(\overline{x})}{2} + \frac{F(\overline{y})}{2} = \inf_{x \in X} F(x)$$

which proves that the minimiser is unique.

Moreover, if $\|\cdot\|_X$ is induced by the scalar product $\langle \cdot, \cdot \rangle_X$, then the minimiser $\overline{x} \in X$ of F has the property that

$$\forall y \in X: \quad 0 = \frac{d}{dt}\bigg|_{t=0} F(\overline{x} + ty) = \sum_{i=1}^n \langle y, \overline{x} - x_i \rangle_X = \left\langle y, \sum_{i=1}^n (\overline{x} - x_i) \right\rangle_X.$$

Consequently,

$$\sum_{i=1}^n (\overline{x} - x_i) = 0 \quad \Rightarrow \quad n\overline{x} = \sum_{i=1}^n x_i \quad \Rightarrow \quad \overline{x} = \sum_{i=1}^n \frac{1}{n} x_i$$

which proves that $\overline{x}$ is in the convex hull of $\{x_1, \ldots, x_n\} \subset X$.

Problem 3.

(a) The distance $d(\cdot,\cdot)$ on the vector space V is induced by a norm if and only if

$$\begin{aligned} &\forall x, y, v \in V: && d(x+v, y+v) = d(x,y),\\ &\forall x, y \in V \quad \forall \lambda \in \mathbb{R}: && d(\lambda x, \lambda y) = |\lambda| d(x,y). \end{aligned}$$

(b) Let $f \in C^0([0,\infty[) \setminus \{0\}$ be supported in $[0,1]$ and let $\lambda > 0$. Then

$$d(\lambda f, 0) = \left(\sum_{n=1}^{\infty} 2^{-n}\right) \frac{\lambda \|f\|_{C^0([0,1])}}{1 + \lambda \|f\|_{C^0([0,1])}} \xrightarrow{\lambda\to\infty} 1$$

which proves that d is not homogeneous and thus not induced by a norm.

Problem 4.

(a) T is not extendable to a bounded linear operator $T\colon c_0 \to \ell^1$. In fact, denoting $e_k = (0,\dots,0,1,0,\dots,0)$, we have for all $k \in \mathbb{N}$

$$\|Te_k\|_{\ell^1} = k-1 = (k-1)\|e_k\|_{\ell^\infty}.$$

(b) Since $(\ell^1)^* \cong \ell^\infty$ and $(c_0)^* \cong \ell^1$ (compare Problem 8.1) we have

$$D_{T^*} = \{y \in \ell^\infty \mid c_c \ni x \mapsto \sum_{n\in\mathbb{N}} y_n (Tx)_n \text{ is continuous}\}.$$

The map $A\colon (c_c, \|\cdot\|_{\ell^\infty}) \to \mathbb{R}$ given by

$$Ax = \sum_{n\in\mathbb{N}} y_n (Tx)_n = \sum_{n=0}^{\infty} y_n n x_{n+1} = \sum_{k=1}^{\infty} y_{k-1}(k-1)x_k$$

is continuous if

$$\sum_{n\in\mathbb{N}} |ny_n| < \infty$$

because

$$|Ax| = \left|\sum_{n\in\mathbb{N}} ny_n x_{n+1}\right| \leq \|x\|_{\ell^\infty} \sum_{n\in\mathbb{N}} |ny_n|.$$

Conversely, if A is continuous, we consider $x^{(N)} = (x_n^{(N)})_{n\in\mathbb{N}} \in c_c$ with $x_n^{(N)} = \frac{y_n}{|y_n|}$ for $n \leq N$ and $x_n^{(N)} = 0$ for $n > N$ to obtain

$$\|A\| = \|A\|\|x^{(N)}\| \geq |Ax^{(N)}| = \left|\sum_{n=0}^{N-1} |ny_n|\right|.$$

Since $N \in \mathbb{N}$ is arbitrary, we conclude

$$\sum_{n\in\mathbb{N}} |ny_n| < \infty.$$

Hence, $D_{T^*} = \{y \in \ell^\infty \mid \sum_{n\in\mathbb{N}} |ny_n| < \infty\}$ and

$$(T^*y)_n = \begin{cases} (n-1)y_{n-1} & (n \geq 1), \\ 0 & (n = 0). \end{cases}$$

(c) The operator is closable. Indeed, suppose $x^{(k)} \in c_c$ for $k \in \mathbb{N}$ satisfy

$$\|x^{(k)}\|_{\ell^\infty} \to 0, \qquad \|Tx^{(k)} - y\|_{\ell^1} \to 0 \qquad (k \to \infty)$$

for some $y \in \ell^1$. For every fixed $n \in \mathbb{N}$ in particular,

$$x_n^{(k)} \to 0, \qquad n x_n^{(k)} \to y_n \qquad (k \to \infty)$$

which implies $y_n = 0$ for all $n \in \mathbb{N}$. Hence, T is closable.

Moreover, by definition,

$$D_{\overline{T}} = \{x \in c_0 \mid \exists (x^{(k)})_{k\in\mathbb{N}} \subset c_c,\ y \in \ell^1 :\ (x^{(k)}, Tx^{(k)}) \to (x, y)\}.$$

Consider $x = (n^{-3})_{n\in\mathbb{N}} \in c_0 \setminus c_c$ and $y = (n^{-2})_{n\in\mathbb{N}} \in \ell^1$. Let $x^{(k)} \in c_c$ be the truncation of x at index k. Then, $x^{(k)} \to x$ in c_0 and

$$\|Tx^{(k)} - y\|_{\ell^1} = \sum_{n=k}^{\infty} n^{-2} \xrightarrow{k\to\infty} 0.$$

Therefore, $x \in D_{\overline{T}}$.

Problem 5.

(a) Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space and let $A\colon H \to H$ be linear, compact and self-adjoint and $A \neq 0$. Then there exist at most countably many eigenvalues $\lambda_k \in \mathbb{R}\setminus\{0\}$ which can accumulate only at $0 \in \mathbb{R}$ and corresponding eigenvectors $e_k \in H$ such that

$$\forall x \in H: \quad Ax = \sum_k \lambda_k \langle x, e_k\rangle e_k.$$

(b) As an orthogonal complement of H'' the subspace $H' \subset H$ is closed and $(H', \langle\cdot,\cdot\rangle)$ is Hilbertean. Let $B_1 \subset H$ be the unit ball in H and $B_1' \subset H'$ the unit ball in H'.

Then, $\overline{A'B_1'} = \overline{AB_1'}$ is compact as closed subset of the compact set $\overline{AB_1}$.

Therefore, $A'\colon H' \to H'$ and analogously $A''\colon H'' \to H''$ are compact operators.

Moreover,

$$\forall x, y \in H': \quad \langle A'x, y\rangle = \langle Ax, y\rangle = \langle x, Ay\rangle = \langle x, A'y\rangle$$

Hence, $A'\colon H' \to H'$ is symmetric and hence self-adjoint being defined on all of H'. Self-adjointness of $A''\colon H'' \to H''$ follows analogously.

(c) The Courant–Fischer characterization of the k-th eigenvalue λ_k of A is

$$\lambda_k = \sup_{\substack{M\subset H,\\ \dim M = k}} \inf_{\substack{x\in M,\\ \|x\|=1}} \langle x, Ax\rangle.$$

(d) By the Courant–Fischer characterization

$$\lambda_1 = \sup_{\substack{x\in H,\\ \|x\|=1}} \langle x, Ax\rangle \geq \sup_{\substack{x'\in H',\\ \|x'\|=1}} \langle x', Ax'\rangle = \sup_{\substack{x'\in H',\\ \|x'\|=1}} \langle x', A'x'\rangle = \lambda_1'.$$

Analogously, $\lambda_1 \geq \lambda_1''$, hence we have $\lambda_1 \geq \max\{\lambda_1', \lambda_1''\}$.

If $e_1 = e_1' + e_1'' \in H' \oplus H''$ is an eigenvector of A to its first eigenvalue $\lambda_1 > 0$, then

$$\begin{aligned} &A'e_1' + A''e_1'' = Ae_1 = \lambda_1 e_1' + \lambda_1 e_1'' \\ \Rightarrow\ &\langle e_1', A'e_1'\rangle = \lambda_1\langle e_1', e_1'\rangle \\ \Rightarrow\ &\langle e_1'', A''e_1''\rangle = \lambda_1\langle e_1'', e_1''\rangle \end{aligned}$$

If $e_1' \neq 0$, then $e_1'/\|e_1'\|$ it is a competitor for λ_1' and we obtain $\lambda_1' \geq \lambda_1$. If $e_1'' \neq 0$, then $e_1''/\|e_1''\|$ it is a competitor for λ_1'' and we obtain $\lambda_1'' \geq \lambda_1$. Since $e_1 \neq 0$, one of these two cases must be true, we conclude $\lambda_1 = \max\{\lambda_1, \lambda_2\}$.

Problem 6.

(a) Let $\Omega \subset \mathbb{R}^n$ be bounded.

A subset $M \subset C^0(\overline{\Omega})$ is relatively sequentially compact, if and only if

$$\sup_{f\in M} \|f\|_{C^0(\overline{\Omega})} < \infty \quad \text{and}$$
$$\forall \varepsilon > 0 \ \ \exists \delta > 0 \ \ \forall x, y \in \Omega \ \ \forall f \in M : \quad \Big(|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon\Big).$$

(b) Let $S \subset C^0([0,1])$ be a linear subspace which is closed with respect to $\|\cdot\|_{C^0}$ and satisfies $S \subset C^1([0,1])$.

Let $(f_k)_{k\in\mathbb{N}}$ be a sequence in S and let $f \in C^0([0,1])$ and $g \in C^1([0,1])$ such that $\|f_k - f\|_{C^0([0,1])} + \|f_k - g\|_{C^1([0,1])} \to 0$ as $k \to \infty$. In particular, $\|f_k - g\|_{C^0([0,1])} \to 0$ as $k \to \infty$. By uniqueness of limits, $f = g \in C^1([0,1])$. Hence, the graph of the inclusion map $S \hookrightarrow C^1([0,1])$ is closed.

(c) Since $S \subset C^0([0,1])$ a closed subspace, $(S, \|\cdot\|_{C^0})$ is a Banach space. Since the inclusion map $S \hookrightarrow C^1([0,1])$ is linear with closed graph by (b), the closed graph theorem states that $S \hookrightarrow C^1([0,1])$ is continuous. In particular,

$$\exists C < \infty \quad \forall f \in S : \quad \|f\|_{C^1([0,1])} \le C\|f\|_{C^0([0,1])}.$$

Let $B_1 \subset S$ be the unit ball in $(S, \|\cdot\|_{C^0})$. Then, $\sup_{f\in B_1} \|f\|_{C^0([0,1])} = 1$ and

$$\forall f \in B_1 : \quad |f(x) - f(y)| \le \|f'\|_{C^0([0,1])}|x-y| \le C|x-y|.$$

Hence, the Arzelà–Ascoli theorem applies and yields that $\overline{B_1} \subset S$ is compact.

Consequently, S is finite dimensional since the closed unit ball would be non-compact otherwise.

Problem 7.

(a) We first observe that $F_{p,N} \subset L^p(\Omega) \subset L^1(\Omega)$, being $|\Omega| < \infty$.

Let $(f_k)_{k\in\mathbb{N}}$ be a sequence in $F_{p,N}$ converging to some f in $L^1(\Omega)$. Up to subsequences, we can assume that $f_k \to f$ pointwise a. e.

If $p < \infty$ then Fatou's lemma gives

$$\int_\Omega |f(x)|^p \,dx = \int_\Omega \lim_{k\to\infty} |f_k(x)|^p \,dx \le \liminf_{k\to\infty} \int_\Omega |f_k(x)|^p \,dx \le N^p,$$

so that $\|f\|_{L^p} \le N$, i. e. $f \in F_{p,N}$.

If $p = \infty$, we can find negligible sets $E', E_k \subset \Omega$ such that $f_k \to f$ everywhere on $\Omega \setminus E'$ and $|f_k| \le N$ everywhere on $\Omega \setminus E_k$. Setting $E := E' \cup (\bigcup_k E_k)$, we get

$$|f(x)| = \lim_{k\to\infty} |f_k(x)| \le N \qquad \forall x \in \Omega \setminus E.$$

Since E is negligible, we infer that $\|f\|_{L^\infty} \le N$, i. e. $f \in F_{\infty,N}$.

(b) We claim that

$$X = \bigcup_{n=1}^{\infty} X_n, \qquad\qquad X_n := X \cap F_{1+\frac{1}{n},n}.$$

Indeed, if $f \in X$ then by hypothesis $f \in L^p(\Omega)$ for some $p > 1$. For n large enough we have $p_n := 1 + \frac{1}{n} < p$ and, by Hölder's inequality,

$$\|f\|_{L^{p_n}}^{p_n} = \int_\Omega |f(x)|^{p_n} \,dx \le \left(\int_\Omega |f(x)|^p \,dx\right)^{\frac{p_n}{p}} |\Omega|^{1-\frac{p_n}{p}}$$

(if $p < \infty$), so that $\|f\|_{L^{p_n}} \le \|f\|_{L^p} |\Omega|^{\frac{1}{p_n}-\frac{1}{p}}$ (which is trivially true also when $p = \infty$). We deduce that $\|f\|_{L^{p_n}} \le \|f\|_{L^p} \max\{|\Omega|, 1\}$. Hence, if n is large enough we get $f \in X_n$.

By (a) the set X_n is closed in X. The vector space X is a Banach space (hence a complete metric space) with the L^1-norm, being X closed in $L^1(\Omega)$. By Baire's category theorem applied to X, there exists n_0 such that X_{n_0} has nonempty interior in X. This means that, for some $f_0 \in X$ and some $r_0 > 0$,

$$X \cap B_{r_0}^{L^1}(f_0) \subset X_{n_0} \subset L^{1+\frac{1}{n_0}}(\Omega).$$

In particular $f_0 \in X \cap L^{1+\frac{1}{n_0}}(\Omega)$, so translating by $-f_0$ we get

$$X \cap B_{r_0}^{L^1}(0) = (X - f_0) \cap B_{r_0}^{L^1}(0) \subset L^{1+\frac{1}{n_0}}(\Omega) - f_0 = L^{1+\frac{1}{n_0}}(\Omega).$$

Thus, choosing $q := 1 + \frac{1}{n_0}$,

$$X = \bigcup_{s>0} \Big(X \cap B^{L^1}_{sr_0}(0) \Big) = \bigcup_{s>0} s \Big(X \cap B^{L^1}_{r_0}(0) \Big) \subset \bigcup_{s>0} sL^q(\Omega) = L^q(\Omega).$$

(c) As already observed, $(X, \|\cdot\|_{L^1})$ and $(L^q(\Omega), \|\cdot\|_{L^q})$ are both Banach spaces. Let us apply the closed graph theorem to the (linear) inclusion map $X \subset L^q(\Omega)$: this map has closed graph because, whenever (f, g) belongs to the closure of the graph, there exists a sequence f_k such that (f_k, f_k) converges to (f, g) in $X \times L^q(\Omega)$, hence up to subsequences f_k converges pointwise a. e. to both f and g, proving that $f = g$ and thus that $(f, g) = (f, f)$ lies in the graph. The closed graph theorem tells us that the inclusion map is continuous. The statement follows by the characterization of continuous linear maps.