8.1. Duality of sequence spaces.

(i) Recall Problem 2.4.

(ii) Follow the proof that $(L^p(\Omega))^*$ is isometrically isomorphic to $L^q(\Omega)$ for $\frac{1}{p}+\frac{1}{q}=1$ (Satz 4.4.1) and construct an isometry $\Psi\colon \ell^1 \to c_0^*$ analogously. To prove surjectivity, show that every functional $f \in c_0^*$ is determined by its values on the elements $e^{(k)} = (e_n^{(k)})_{n\in\mathbb{N}} \in c_0$, where $e^{(k)} = (0,\ldots,0,1,0,\ldots)$ has the 1 at k-th position.

(iii) Find a bijective linear map $T\colon c \to c_0$ and compose functionals in c_0^* with T to obtain functionals in c^*.

8.2. A result by Lions-Stampacchia. Check that the Lax-Milgram Theorem applies to $a(\cdot,\cdot)$ and prove that there exists a unique $x_0 \in H$ with $J(x) = a(x-x_0, x-x_0) - a(x_0,x_0)$.

For the first inequality, show that $(H, a(\cdot,\cdot))$ is a Hilbert space and apply the result of Problem 7.6 (ii) in $(H, a(\cdot,\cdot))$. Note that convexity of K is used here.

For the second inequality, combine and exploit convexity of K and the minimality property of y_0.

8.3. Projection to convex sets. Use the second inequality of Problem 8.2 (for $a(\cdot,\cdot) = (\cdot,\cdot)_H$ with $y_0 = Px_0$) to prove that if $y \in K$, then $(Px_0 - x_0, y - Px_0)_H \geq 0$ for any $x_0 \in H$. You can use this twice in (i) and once in (ii).

8.4. Strict convexity.

(i) Suppose two different elements in X^* have the property in question. Divide by $\|x\|_X^2$ such that everything has norm 1 and apply strict convexity of X^* with $\lambda = 1/2$.

(ii) You can find a finite dimensional example.

8.5. Uniform convexity.

(i) Recall the parallelogram identity (cf. Problem 1.2).

(ii) The supremum norm is usually a good counterexample for strict and uniform convexity.

8.6. Functional on the span of a sequence. Read the inequality in (ii) as $|\tilde{\ell}(y)| \leq \gamma \|y\|_X$ for some functional $\tilde{\ell}$ defined on a suitable subspace of X which can be extended to X using the Hahn-Banach Theorem.