9.1. Minkowski functional ⚙. Recall the definition of the Minkowski functional

$$\begin{aligned} p\colon X &\to \mathbb{R} \\ x &\mapsto \inf\{\lambda > 0 \mid \tfrac{1}{\lambda}x \in Q\}. \end{aligned}$$

Define the set $\Upsilon \subset X^*$ by using p and prove the two inclusions. By showing $p(x) < 1$ for $x \in Q$, one inclusion follows directly. Prove the other inclusion indirectly.

(The set $\Upsilon \subset X^*$ is not required to be open.)

9.2. Extremal subsets ✎.

(i) Show that E is a subset of the boundary ∂K. Derive a contradiction to $E \subset \partial K$ under the assumption that E is not closed.

(ii) Draw a picture.

(iii) $K \setminus M$ being not convex contradicts the definition of extremal subset. Notice where convexity of K is used.

(iv) Have you tried intervals?

(v) If $y \in K$ is an extremal point of K, then $\{y\} \subset K$ is an extremal subset of K.

9.3. Weak sequential continuity of linear operators ✎. Prove (ii) $\Rightarrow$ (i) by contradiction and use that weakly convergent sequences must be bounded (Satz 4.6.1).

9.4. Weak convergence in finite dimensions ✎. Recall that all norms are equivalent in finite dimensions.

9.5. Weak convergence in Hilbert spaces ✎.

(i) $x_n \xrightarrow{\text{w}} x$ implies $(x, x_n)_H \to (x, x)_H$.

(ii) Weakly convergent sequences are bounded.

(iii) Recall Bessel's inequality.

(iv) Use (iii).

(v) Use (iii).

9.6. Sequential closure ✎.

(i) Argue by contradiction.

(ii) Aim at finding a set $\Omega \subset \ell^2$ such that $(0) := (0, 0, \dots) \in \overline{\Omega}_{\text{w}}$ but no sequence in Ω converges weakly to zero: $(0) \notin \overline{\Omega}_{\text{w-seq}}$. Notice that such $\Omega \subset \ell^2$ must be unbounded. Recall what $(0) \in \overline{\Omega}_{\text{w}}$ and $(0) \notin \overline{\Omega}_{\text{w-seq}}$ both mean by definition.

9.7. Convex hull ⚙.

(i) First note that arbitrary subsets $\Omega \subset X$ satisfy the inclusions

$$\Omega \subset \overline{\Omega} \subset \overline{\Omega}_{\text{w-seq}} \subset \overline{\Omega}_{\text{w}},$$

where $\overline{\Omega}$ denotes the closure in the norm-topology, $\overline{\Omega}_{\text{w-seq}}$ the weak-sequential closure and $\overline{\Omega}_{\text{w}}$ the closure in the weak topology. Mazur's Lemma is based on the fact that for convex sets, the closure with respect to the norm-topology agrees with the closure in the weak topology (Satz 4.6.2).

(ii) Show first that

$$\operatorname{conv}(A \cup B) = \bigcup_{\substack{s,t \geq 0 \\ s+t=1}} (sA + tB)$$

and then argue that the right hand side is compact.